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A second-order accurate numerical method for the space–time tempered fractional diffusion-wave equation

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A B S T R A C T

This paper focuses on providing the high order algorithms for the space–time tempered fractional diffusion-wave equation. The designed schemes are unconditionally stable and have the global truncation error $\mathcal{O}(\tau^2 + h^2)$, being theoretically proved and numerically verified.

1. Introduction

Tempered anomalous diffusion describes the very slow transition from anomalous to normal diffusion, and it has many applications in physical, biological, and chemical processes. In some cases, the transition even does not appear at all in the observation time because of the finite lifespan of the particles or the finite observation time of the experimentalist [1]. This paper considers the numerical methods for the macroscopic model describing this kind of phenomena. More concretely, we study a second-order accurate numerical method in both space and time for the integro-differential equation whose prototype is, for $1 < \alpha, \gamma \le 2, \lambda \ge 0$,

$$\frac{\partial}{\partial t}u(x,t) = I_t^{\gamma-1,\lambda}\nabla_x^\alpha u(x,t) = \frac{1}{\Gamma(\gamma-1)}\int_0^t (t-\tau)^{\gamma-2}e^{-\lambda(t-\tau)}\nabla_x^\alpha u(x,\tau)d\tau, \tag{1.1}$$

with the initial condition $u(x,0) = u_0(x)$, $x \in \Omega = (a,b)$ and the homogeneous Dirichlet boundary conditions, characterizing the propagation of wave with the tempered power law decay. Here the tempered

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fractional integral $I_t^{\beta,\lambda}$ with $\beta = \gamma - 1 > 0$ is defined as [2,3]

$$I_t^{\beta,\lambda}u(x,t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} e^{-\lambda(t-\tau)} u(x,\tau)d\tau, \qquad t > 0. \tag{1.2}$$

The Riesz fractional derivative with $\alpha \in (1,2)$, is defined as [4]

$$\nabla_x^\alpha u(x,t) = -\kappa_\alpha \left({}_aD_x^\alpha + {}_xD_b^\alpha \right) u(x,t) \ \text{ with } \ \kappa_\alpha = \frac{1}{2\cos(\alpha\pi/2)}, \tag{1.3}$$

$${}_aD_x^\alpha u(x,t) = \frac{1}{\Gamma(2-\alpha)} \frac{\partial^2}{\partial x^2} \int_a^x (x-\xi)^{1-\alpha} u(\xi,t)d\xi, \quad {}_xD_b^\alpha u(x,t) = \frac{1}{\Gamma(2-\alpha)} \frac{\partial^2}{\partial x^2} \int_x^b (\xi-x)^{1-\alpha} u(\xi,t)d\xi.$$

It can be noted that, if $\lambda = 0$, (1.1) reduces to the following space–time fractional diffusion-wave equation [5],

$${}^cD_t^\gamma u(x,t) = \nabla_x^\alpha u(x,t) \ \text{ for } \ 1 < \alpha, \gamma \le 2.$$

Numerical methods for the time discretization of (1.1) with $\lambda = 0, \alpha = 2$ have been proposed by various authors [6–9]. For example, Cuesta (2006) et al. derive the second-order error bounds of the time discretization in a Banach space with ∇_x^2 as a sectorial operator [6]; and Yang (2014) et al. obtain the second-order convergence schemes with $1 \le \gamma \le 1.71832$ [9]. McLean and Mustapha (2007) study the Crank–Nicolson scheme for the time discretization with the non-uniform grid [10].

The space–time tempered fractional diffusion-wave equation of (1.1) with $\lambda = 0$ are discussed in [11,12] and its numerical solutions are designed in [13,14]. However, it seems that achieving a second-order accurate scheme for (1.1) is not an easy task. This paper focuses on providing effective and highly accurate numerical algorithms for (1.1). The designed schemes are unconditionally stable and have the global truncation error $\mathcal{O}(\tau^2 + h^2)$, being theoretically proved and numerically verified. It can be easily extended to the problems discussed in [9,13,14].

The rest of the paper is organized as follows. The next section proposes our second-order accurate scheme for (1.1). In Section 3, we carry out a detailed stability and convergence analysis with the second order accuracy in both time and space directions for the derived schemes. To show the effectiveness of the schemes, we perform the numerical experiments to verify the theoretical results in Section 4. The paper is concluded with some remarks in the last section.

2. High order schemes for the space–time tempered fractional diffusion-wave equation

Let the mesh points $x_i = a + ih$ for $i = 0, 1, \ldots, M$, and $t_n = n\tau$ for $n = 0, 1, \ldots, N$, where $h = (b-a)/M$ and $\tau = T/N$ are the uniform space stepsize and time steplength, respectively. Denote u_i^n as the numerical approximation to $u(x_i, t_n)$. Here, we utilize the second-order formula [15,16] to approximate the Riesz fractional derivative (1.3), that is

$$\nabla_x^\alpha u(x,t)|_{x=x_i} = -\frac{\kappa_\alpha}{\Gamma(4-\alpha)h^\alpha} \sum_{j=1}^{M-1} w_{i,j}^\alpha u(x_j,t) + \mathcal{O}(h^2) \tag{2.1}$$

with $i = 1, \ldots, M-1$, where

$$w_{i,j}^\alpha = \begin{cases} w_{i-j+1}^\alpha, & j < i-1, \\ w_0^\alpha + w_2^\alpha, & j = i-1, \\ 2w_1^\alpha, & j = i, \\ w_0^\alpha + w_2^\alpha, & j = i+1, \\ w_{j-i+1}^\alpha, & j > i+1, \end{cases} \quad \text{and} \quad w_m^\alpha = \begin{cases} 1, & m = 0, \\ -4 + 2^{3-\alpha}, & m = 1, \\ 6 - 2^{5-\alpha} + 3^{3-\alpha}, & m = 2, \\ (m+1)^{3-\alpha} - 4m^{3-\alpha} + 6(m-1)^{3-\alpha} & \\ \quad - 4(m-2)^{3-\alpha} + (m-3)^{3-\alpha}, & m \ge 3. \end{cases}$$