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Robust fixed stress splitting for Biot's equations in heterogeneous media

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ABSTRACT

We study the iterative solution of coupled flow and geomechanics in heterogeneous porous media, modeled by a three-field formulation of the linearized Biot's equations. We propose and analyze a variant of the widely used Fixed Stress Splitting method applied to heterogeneous media. As spatial discretization, we employ linear Galerkin finite elements for mechanics and mixed finite elements (lowest order Raviart–Thomas elements) for flow. Additionally, we use implicit Euler time discretization. The proposed scheme is shown to be globally convergent with optimal theoretical convergence rates. The convergence is rigorously shown in energy norms employing a new technique. Furthermore, numerical results demonstrate robust iteration counts with respect to the full range of Lamé parameters for homogeneous and heterogeneous media. Being in accordance with the theoretical results, the iteration count is hardly influenced by the degree of heterogeneities.

1. Introduction

The coupling of mechanics and flow in porous media is relevant for many applications ranging from environmental engineering to biomedical engineering. The simplest model of real applied importance is the quasi-static linearized Biot system, applicable for infinitesimally deforming, fully saturated porous media. Existence, uniqueness and regularity for Biot's equations have been investigated first by Showalter [1].

There are two approaches currently employed for solving Biot's equations. They are referred to as fully-implicit and iterative coupling [2]. The fully-implicit approach involves solving the fully coupled system of governing equations simultaneously, providing the benefit of unconditional stability. It requires advanced and efficient preconditioners. For this purpose, (Schur complement based) block preconditioners appear to be a sound choice [3–6]. The iterative coupling approach involves the sequential-implicit solution of flow and

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mechanics using the latest solution information, iterating the procedure at each time step until convergence. The sequential-implicit approach offers greater flexibility in code design than the fully-implicit approach. On the other hand, being equivalent to a preconditioned Richardson method [7], sequential-implicit approaches also provide a basis to design efficient block preconditioners for the fully-implicit approach [8,9]. Among iterative coupling schemes, the widely used Fixed Stress Splitting method has been rigorously shown to be unconditionally stable in the sense of a Von Neumann analysis [10] and globally convergent [11], when considering slightly compressible flow in a homogeneous porous medium.

The new contributions of this work are:

- We prove global, linear convergence in energy norms of the Fixed Stress Splitting method applied to the fully discretized three-field formulation of Biot's equations for heterogeneous media, where linear finite elements are employed for mechanics, mixed finite elements (lowest order Raviart–Thomas elements) are employed for flow, and backward Euler time discretization is applied.
- We propose a new, optimized tuning parameter for heterogeneous media.

In the case of homogeneous media, the results are in consistency with previous numerical studies, cf., e.g., [12]. To the best of our knowledge, this is the first time the convergence of the Fixed Stress Splitting method is rigorously shown for energy norms and considering heterogeneous media.

2. Mathematical model — Biot's equations

We consider the quasi-static Biot's equations [13,14], modeling a linearly elastic porous medium $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, saturated with a slightly compressible fluid. On the space–time domain $\Omega \times (0,T)$, the governing equations read

$$-\boldsymbol{\nabla}\cdot[2\mu\boldsymbol{\varepsilon}(\boldsymbol{u})+\lambda\boldsymbol{\nabla}\cdot\boldsymbol{u}]+\alpha\boldsymbol{\nabla}p=\boldsymbol{f}, \qquad \partial_t\left(\frac{p}{M}+\alpha\boldsymbol{\nabla}\cdot\boldsymbol{u}\right)+\boldsymbol{\nabla}\cdot\boldsymbol{w}=S_f, \qquad \boldsymbol{K}^{-1}\boldsymbol{w}+\boldsymbol{\nabla}p=\rho_f\boldsymbol{g}. \quad (1)$$

Here, $\boldsymbol{u}$ is the displacement, p is the fluid pressure, $\boldsymbol{w}$ is the Darcy flux, $\boldsymbol{\varepsilon}(\boldsymbol{u}) = 0.5(\boldsymbol{\nabla}\boldsymbol{u} + \boldsymbol{\nabla}\boldsymbol{u}^\top)$ is the linearized strain tensor, μ, λ are the Lamé parameters, α is the Biot coefficient, M is the Biot modulus, ρ_f is the fluid density, $\boldsymbol{K}$ is the permeability tensor divided by fluid viscosity, $\boldsymbol{g}$ is the gravity vector, and S_f is a volume source term. For simplicity, we assume homogeneous boundary $\boldsymbol{u} = \boldsymbol{0}$, $p = 0$ on $\partial\Omega \times [0,T]$ and initial conditions $\boldsymbol{u} = \boldsymbol{u}_0$, $p = p_0$ in $\Omega \times \{0\}$. We make the following assumptions on the effective coefficients:

(A1) Let $\rho_f \in \mathbb{R}$, $\boldsymbol{g} \in \mathbb{R}^d$ be constant.
(A2) Let $M, \alpha, \mu, \lambda \in L^\infty(\Omega)$ be positive, uniformly bounded, with the lower bound strictly positive.
(A3) Let $\boldsymbol{K} \in L^\infty(\Omega)^{d\times d}$ be a symmetric matrix, which is constant in time and has uniformly bounded eigenvalues, i.e., there exist constants $k_m, k_M \in \mathbb{R}$, satisfying for all $\boldsymbol{x} \in \Omega$ and for all $\boldsymbol{z} \in \mathbb{R}^d \setminus \{\boldsymbol{0}\}$

$$0 < k_m \boldsymbol{z}^\top\boldsymbol{z} \le \boldsymbol{z}^\top\boldsymbol{K}(x)\boldsymbol{z} \le k_M\boldsymbol{z}^\top\boldsymbol{z} < \infty.$$

Below, we consider a numerical approximation of the weak solution of Biot's equations as described above.

3. Fixed stress splitting for the fully discretized system

Let $\mathcal{T}_h$ be a regular decomposition of mesh size h of the domain Ω. Furthermore, let $0 = t_0 < t_1 < \cdots < t_N = T$, $N \in \mathbb{N}$, define a partition of the time interval $(0,T)$ with constant time step size $\tau = t_{k+1} - t_k$, $k \ge 0$. In order to discretize Biot's equations in space, we use linear, constant and lowest order Raviart–Thomas