Applied Mathematics Letters 68 (2017) 175–180

Contents lists available at ScienceDirect

Applied Mathematics Letters

www.elsevier.com/locate/aml

Water-wave profiles from pressure measurements: Extensions

Vishal Vasan[a,*], Katie L. Oliveras[b]

[a] *International Centre for Theoretical Sciences, Bengaluru, 560089, India*
[b] *Department of Mathematics, Seattle University, Seattle, WA 98122, United States*

ARTICLE INFO

Article history:
Received 24 August 2016
Received in revised form 23 January 2017
Accepted 24 January 2017
Available online 28 January 2017

Keywords:
Water waves
Pressure
Recovery
Measurement

ABSTRACT

In this letter, we present a method to reconstruct the surface elevation of an inviscid traveling gravity wave in two dimensions using the pressure measured at any depth. In order to achieve this reconstruction, we derive a map from the pressure at an arbitrary depth $p(x, z_0)$ to the horizontal velocity at the bottom of the fluid domain. Once this map is established, we use the result presented in Oliveras et al. (2012) to find the free-surface η. In addition to the fully nonlinear relationship, we present a "heuristic" asymptotic formula that does not involve the wave speed c.

1. Introduction

Pressure transducers are a common device for measuring surface water waves in both the laboratory and the field. However, the inversion from the time series of pressure to the time series of surface displacement is nontrivial. The relationship between pressure and surface displacement, based on the Stokes boundary value problem [1], is nonlinear and involves fluid velocities. In practice, however, many of the reconstruction methods rely on a linear analysis of the fluid equations. There have been a number of field and laboratory studies to determine if linear theory provides an adequate relationship between pressure measurements and surface displacement, and some disagreement exists [2]. On the other hand, attempts to add nonlinear effects include empirical factors [3–5] as well as perturbative calculations using a Stokes expansion [6,7]. Recently, Oliveras et al. [8] proposed a fully nonlinear formulation that depends on a reformulation of the Stokes boundary value problem rather than beginning with a perturbation expansion in the surface displacement and velocity potential. This formulation represents an exact expression for the surface displacement of a traveling wave that requires the numerical solution of a nonlocal, nonlinear equation.

While the formulation of [8] represented a significant advancement in relating the pressure to the shape of the free surface, it does require that the pressure be measured at the bottom of the fluid domain. An

* Corresponding author.
E-mail address: vishal.vasan@icts.res.in (V. Vasan).

http://dx.doi.org/10.1016/j.aml.2017.01.017

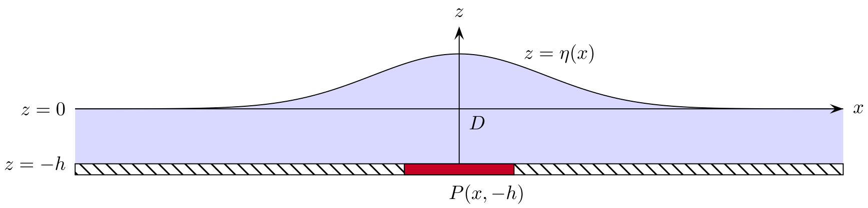

Fig. 1. The fluid domain D for the water wave problem. A pressure sensor is indicated at the bottom. In the current work we handle the case when the sensor is at some height above the bottom.

important dynamical effect present in water waves is the attenuation of the fluid velocity field with depth. This has important implications on wave reconstruction as the fluid itself acts like a low pass filter and hence high frequency information is removed from the pressure record. Indeed as Kinsman [9] notes, one should not "go around making general statements about the sea surface on the strength of the bottom pressure records", a remark made in the context of linear theory. To overcome this loss of data, practitioners employ pressure gauges affixed at heights above the sea bottom [10–13]. Thus wave reconstruction from pressure measurements located above the sea bottom is a practical requirement for both laboratory and field experiments.

Bottom pressure gauges may also pose practical challenges associated with installation of the pressure transducer on the bed. Furthermore, the pressure signal can be modulated by the elastic response of the sea bottom, as typically happens for soft sandy bottom surfaces [14]. Bottom-bed installations may suffer from scour which can affect the local fluid pattern and associated pressures. In order to eliminate problems associated with surface-piercing gauges, the laboratory experiments described in [15] employ pressure transducers 12 feet above the bottom bed in water of average depth 68 feet. Hence measurements were recorded well within the bulk of the fluid. In this short communication, we extend the formulation given in [8] to allow for arbitrary depth of the pressure transducers. We also show how to accordingly modify the heuristic formulation of [8].

2. Reconstructing

Consider Euler's equations describing the dynamics of a traveling wave at the free surface of an ideal incompressible irrotational fluid in two dimensions (with a one-dimensional surface):

$$\phi_{xx} + \phi_{zz} = 0, \qquad (x, z) \in D, \tag{1}$$

$$\phi_z = 0, \qquad z = -h, \tag{2}$$

$$-c\eta_x + \eta_x \phi_x = \phi_z, \qquad z = \eta(x), \tag{3}$$

$$-c\phi_x + \frac{1}{2}\left(\phi_x^2 + \phi_z^2\right) + g\eta = 0, \qquad z = \eta(x), \tag{4}$$

where $\phi(x, z)$ represents the velocity potential of the fluid with surface elevation profile $\eta(x)$ and c is the wave speed. A solution to the above set of equations consists of finding a potential ϕ that satisfies Laplace's equation inside the fluid domain D, sketched in Fig. 1, as well as finding the graph of the free surface η. The problem may be posed on the whole line $x \in \mathbb{R}$, assuming the velocities (and surface profile) decay at infinity, or may also be posed using periodic boundary conditions. Existence of solutions (and properties thereof) was proved in [16–20].

The goal is to relate the pressure measured at a sensor $P(x, -h)$, which is also sketched in Fig. 1, located on the bottom boundary of the fluid, to the surface elevation profile $\eta(x)$. As stated, Eqs. (1)–(4) do not