Applied Mathematics Letters 69 (2017) 29–34

Contents lists available at ScienceDirect

Applied Mathematics Letters

www.elsevier.com/locate/aml

Analytical properties of nonlinear dislocation equation

Nikolay A. Kudryashov

National Research Nuclear University MEPhI (Moscow Engineering Physics Institute), 31 Kashirskoe Shosse, 115409 Moscow, Russian Federation

ARTICLE INFO

Article history:
Received 17 December 2016
Received in revised form 18 January 2017
Accepted 18 January 2017
Available online 13 February 2017

Keywords:
Dislocation
Frenkel–Kontorova model
Fermi–Pasta–Ulam model
Nonlinear dislocation equation
Exact solution

ABSTRACT

Continuum model corresponding to the generalization of both the Fermi–Pasta–Ulam and the Frenkel–Kontorova models is considered. This generalized model can be used for the description of nonlinear dislocation waves in the crystal lattice. Using the Painlevé test we analyze the integrability of this equation. We find that there exists an integrable case of the partial differential equation for nonlinear dislocations. Exact solutions of nonlinear dislocation equation are presented.

1. Introduction

Let us consider the following dynamical system:

$$m\,\frac{d^2 y_i}{dt^2} = F_{i+1,i} - F_{i,i-1} - f_0\,\sin\left(\frac{2\,\pi\,y_i}{a}\right), \qquad (i = 1, \ldots, N), \tag{1}$$

where y_i measures the displacement of the i-th mass from equilibrium in time t, the force $F_{i+1,i}$ describes the nonlinear interaction between atoms in the case of dislocations in the crystal lattice in case of dislocations

$$F_{i+1,i} = \gamma\,(y_{i+1} - y_i) + \alpha\,(y_{i+1} - y_i)^2 + \beta\,(y_{i+1} - y_i)^3, \tag{2}$$

and f_0, a, γ, α, β are constant parameters of system (1).

The system of Eqs. (1) is the generalization of some well-known dynamical systems. At $\alpha = 0$ and $\beta = 0$ the system of Eqs. (1) is the mathematical model introduced by Frenkel and Kontorova for the description of dislocations in the rigid body [1]. In this model it was suggested that the influence of atoms in the crystal is taken into account by term $f_0\,\sin\frac{2\,\pi\,y_i}{a}$ but the atoms in case of dislocations interact by means of linear

E-mail address: nakudr@gmail.com.

http://dx.doi.org/10.1016/j.aml.2017.01.015
0893-9659/

low. Assuming that $N \to \infty$ and $h \to 0$ where h is the distance between atoms, we can get the Sine–Gordon equation.

In case of $f = 0$ and $\beta = 0$ system of Eqs. (1) is the well-known Fermi–Pasta–Ulam model [2] which was studied many times [3,4]. It is known that the Fermi–Pasta–Ulam model is transformed at $N \to \infty$ and $h \to 0$ to the Korteweg–de Vries equation [5].

The main result of work [5] was the introduction of solitons as solutions of the Korteweg–de Vries equation. It was shown in 1967 that the Cauchy problem for this equation can be solved by the Inverse Scattering transform [6].

Assuming $f_0 = 0$, $\alpha \neq 0$ and $\beta \neq 0$ at $N \to \infty$ and $h \to 0$ one can find the modified Korteweg–de Vries equation for the description of nonlinear waves.

In papers [7,8] the authors took into account high order terms in the Taylor series for the description of nonlinear waves in the Fermi–Pasta–Ulam model assuming that $\alpha \neq 0$ and $\beta \neq 0$ and did not obtain nonlinear nonintegrable differential equations in mass chain.

These facts bring up the question, whether the situation is similar for Eq. (1) at $f_0 \neq 0$, $\alpha \neq 0$ and $\beta \neq 0$. In other words we assume that the interaction between dislocations in crystal is described by means of nonlinear low at $\alpha \neq 0$ and $\beta \neq 0$.

The aim of this paper is to derive the nonlinear partial differential equation corresponding to dynamical system (1) and to study the properties of this equation.

The rest of this work is organized as follows. In Section 2 we derive the fourth-order partial differential equation for the description of the nonlinear dislocation waves governed at $N \to \infty$ and $h \to 0$ by system (1). In Section 3 we apply the Painlevé test to study analytical properties of the nonlinear equation. In Section 4 we present exact solutions for the nonlinear partial differential equation corresponding to dynamical system (1). In Section 5 we briefly discuss the results of this work.

2. Derivation of the equation for the description of nonlinear dislocations

Let us assume that the suggestions $N \to \infty$ and $h \to 0$ are carried out for dynamical systems (1). In this case we can use the continuous limit approximation in dynamical system (1). Taking into consideration the expansion of mass $y_{i\pm1}$ from equilibrium position in the Taylor series up to h^4, we obtain

$$y_{i\pm1} = y_i \,\pm\, h\, y_{i,x} + \frac{h^2}{2}\, y_{i,xx} \,\pm\, \frac{h^3}{6}\, y_{i,xxx} \,+\, \frac{h^4}{24}\, y_{i,xxxx} \,+ \cdots . \tag{3}$$

Substituting expansion (3) into Eq. (1) and using the expressions with order up to h^4 inclusively, we obtain the nonlinear fourth-order partial differential equation in the form:

$$m\, y_{tt} = k\, h^2\, y_{xx} + 2\alpha h^3 y_x\, y_{xx} + 3\beta h^4 y_x^2\, y_{xx} + \frac{k h^4}{12} y_{xxxx} - f_0 \sin\left(\frac{2\pi y}{a}\right). \tag{4}$$

Using new parameters and a new variable in the form:

$$\begin{gathered} c = \frac{h\sqrt{k}}{\sqrt{m}}, \quad \varepsilon = \frac{a\, h}{\pi\, k}, \quad \beta' = \frac{3\,\beta\, h\, a}{4\,\pi}, \\ v(x,t) = \frac{2\,\pi}{a}\, y(x,t), \quad \gamma' = \frac{\pi\, k\, h}{12\, a}, \quad \delta = \frac{2\,\pi^2\, f_0}{a^2\, h^3}. \end{gathered} \tag{5}$$

we have that Eq. (4) can be written as the following

$$v_{tt} = c^2\, v_{xx} + \alpha\, c^2\, \varepsilon\, v_x\, v_{xx} + \beta'\, c^2\, \varepsilon\, v_x^2\, v_{xx} + \gamma'\, c^2\, \varepsilon\, v_{xxxx} - \delta\, c^2\, \varepsilon\, \sin v. \tag{6}$$