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Norm continuity of solution semigroups of a class of neutral functional differential equations with distributed delay

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ABSTRACT

In this note, we shall consider the norm continuity of a class of solution semigroups associated with linear functional differential equations of neutral type with time lag $r > 0$ in Hilbert spaces. The norm continuity plays an important role in the analysis of asymptotic stability of the system under consideration by means of spectrum approaches. We shall show that for a square integrable neutral delay term and an unbounded infinitesimal generator A multiplied by a square integrable weight function in the distributed delay term, the associated solution semigroup of the system is norm continuous at every $t > r$.

1. Introduction

For any Banach spaces X and Y, we always denote by $\mathscr{L}(X,Y)$ the space of all bounded, linear operators from X to Y. If $X = Y$, we simply write $\mathscr{L}(X,X)$ by $\mathscr{L}(X)$. Let V be a separable Hilbert space and V^* be its dual space. Suppose that $a : V \times V \to \mathbb{R}$ is a bounded bilinear form satisfying the so-called Gårding's inequality $a(x,x) \le -\alpha\|x\|_V^2$, $x \in V$, for some constant $\alpha > 0$. Let A be a linear operator defined by this form through $a(x,y) = \langle x, Ay\rangle_{V,V^*}, x, y \in V$, where $\langle\cdot,\cdot\rangle_{V,V^*}$ is the duality pair between V and V^*. Then $A \in \mathscr{L}(V, V^*)$ and A generates a C_0-semigroup e^{tA}, $t \ge 0$, on V^* such that $e^{tA} : V^* \to V$ for each $t > 0$. We introduce the Lions interpolation Hilbert space (see Tanabe [1]) $H = (V, V^*)_{1/2,2}$ between V and V^*, which is given by $H = \{x \in V^* : \int_0^\infty \|Ae^{tA}x\|_{V^*}^2 dt < \infty\}$ with its inner product

$$\langle x, y\rangle_H := \langle x, y\rangle_{V^*} + \int_0^\infty \langle Ae^{tA}x, Ae^{tA}y\rangle_{V^*} dt, x,\ y \in H. \tag{1.1}$$

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We identify the dual H^* of H with H, then it is easy to see that $V \hookrightarrow H = H^* \hookrightarrow V^*$ where the embedding $\hookrightarrow$ is dense and continuous with $\|x\|_H^2 \le \nu\|x\|_V^2$, $x \in V$, for some constant $\nu > 0$. Moreover, the semigroup e^{tA}, $t \ge 0$, generated by A is bounded and analytic on both H and V^* such that

$$\|e^{tA}\|_{\mathscr{L}(H)} \le 1, \|e^{tA}\|_{\mathscr{L}(V^*)} \le C, t \ge 0, \tag{1.2}$$

for some constant $C > 0$, and for any $T \ge 0$, there is a continuous embedding

$$L^2([0,T],V) \cap W^{1,2}([0,T],V^*) \subset C([0,T],H) \tag{1.3}$$

where $W^{1,2}([0,T],V^*)$ is the standard Sobolev space (see [2]).

Let $r > 0$ and $\mathcal{H} = H \times L^2([-r,0],V)$. Consider a retarded functional differential equation with distributed delay,

$$\begin{cases} dy(t) = Ay(t)dt + \displaystyle\int_{-r}^{0} \beta(\theta)Ay(t+\theta)d\theta, t \ge 0, \\ y(0) = \phi_0,\ y(\theta) = \phi_1(\theta),\ \theta \in [-r,0],\ \phi = (\phi_0,\phi_1) \in \mathcal{H}, \end{cases} \tag{1.4}$$

where $\beta \in L^2([-r,0],\mathbb{C})$ and $\mathbb{C}$ is the space of all complex numbers. Note that since $\phi_1 \in L^2([-r,0],V)$, it generally does not make sense to talk about $\phi_0 = \phi_1(0)$ unless ϕ_1 satisfies more regular properties, e.g., ϕ_1 is a continuous function. As is well known, for any $\phi \in \mathcal{H}$ there exists a unique strong solution $y = y(t,\phi)$, $t \ge -r$, to (1.4). In particular, one can introduce in association with this solution a C_0-semigroup $\mathcal{S}(t)$, $t \ge 0$, on $\mathcal{H}$ by $\mathcal{S}(t)\phi = (y(t,\phi), y_t(\phi))$ for any $\phi \in \mathcal{H}$, where $y_t(\phi)(\theta) := y(t+\theta,\phi)$, $\theta \in [-r,0]$. As regards norm continuity of $\mathcal{S}(\cdot)$, i.e., $\mathcal{S}(\cdot) : [0,\infty) \to \mathscr{L}(\mathcal{H})$ is continuous in the uniform operator topology, Di Blasio et al. [3] have proved that if the weight function $\beta(\cdot)$ in the distributed delay term of (1.4) satisfies $\beta(\cdot) \in W^{1,2}([-r,0],\mathbb{C})$, then the associated solution semigroup $\mathcal{S}(t)$, $t \ge 0$, is a differentiable (thus, norm continuous) semigroup for $t > r$. Subsequently, Jeong in [4] has shown that if $\beta(\cdot)$ is Hölder continuous, i.e., for any θ, $\tau \in [-r,0]$, $|\beta(\theta) - \beta(\tau)| \le C|\theta-\tau|^\kappa$, $C > 0$, $\kappa \in (0,1]$, then the solution semigroup $\mathcal{S}(t)$ is norm continuous for $t > 3r$. In 2002, Mastinšek [5] further improved their results to obtain that when $\beta(\cdot) \in L^2([-r,0],\mathbb{C})$, the solution semigroup $\mathcal{S}(t)$, $t \ge 0$, is norm continuous for $t > r$.

In this work, we shall generalize the above case to consider the norm continuity of a class of neutral functional differential equations of the form,

$$\begin{cases} d\Big(y(t) - \displaystyle\int_{-r}^{0} D(\theta)y(t+\theta)d\theta\Big) = A\Big(y(t) - \displaystyle\int_{-r}^{0} D(\theta)y(t+\theta)d\theta\Big)dt + \displaystyle\int_{-r}^{0} \beta(\theta)Ay(t+\theta)d\theta dt, t \ge 0, \\ y(0) = \phi_0 + \displaystyle\int_{-r}^{0} D(\theta)\phi_1(\theta)d\theta,\ y(\theta) = \phi_1(\theta),\ \theta \in [-r,0],\ \phi = (\phi_0,\phi_1) \in \mathcal{H}, \end{cases} \tag{1.5}$$

where $D(\cdot) \in L^2([-r,0],\mathscr{L}(V))$ and $\beta(\cdot) \in L^2([-r,0],\mathbb{C})$. It was shown in Liu [6] that for each $\phi = (\phi_0,\phi_1) \in \mathcal{H}$, there exists a unique strong solution $y = y(t,\phi)$, $t \ge 0$, to Eq. (1.5) and for any $T \ge 0$, the solution y satisfies

$$\|y\|_{L^2([0,T],V)} + \|y\|_{W^{1,2}([0,T],V^*)} \le C(T)\|\phi\|_{\mathcal{H}}, \tag{1.6}$$

for some number $C(T) > 0$ which depends on $T \ge 0$. In association with this solution y to (1.5), it was further shown (see [6]) that one can introduce a C_0-semigroup $\mathcal{S}(t)$, $t \ge 0$, of (1.5) on $\mathcal{H}$ given by

$$\mathcal{S}(t)\phi = \Big(y(t,\phi) - \int_{-r}^{0} D(\theta)y(t+\theta,\phi)d\theta, y_t(\phi)\Big), \quad t \ge 0, \phi \in \mathcal{H}. \tag{1.7}$$

In this short note, we shall establish the following result which will play an important role in the stability analysis of the system (1.5) (see, e.g., [3]).