Applied Mathematics Letters 69 (2017) 49–54

Contents lists available at ScienceDirect

Applied Mathematics Letters

www.elsevier.com/locate/aml

A homotopy method for solving multilinear systems with M-tensors

Lixing Han

Department of Mathematics, University of Michigan-Flint, Flint, MI 48502, USA

ARTICLE INFO

Article history:
Received 23 December 2016
Received in revised form 27 January 2017
Accepted 27 January 2017
Available online 14 February 2017

Keywords:
M-tensor
Multilinear system
Homotopy method

ABSTRACT

Multilinear systems of equations arise in various applications, such as numerical partial differential equations, data mining, and tensor complementarity problems. In this paper, we propose a homotopy method for finding the unique positive solution to a multilinear system with a nonsingular M-tensor and a positive right side vector. We analyze the method and prove its convergence to the desired solution. We report some numerical results based on an implementation of the proposed method using a prediction–correction approach for path following.

1. Introduction

Let $\mathbb{R}$ and $\mathbb{C}$ be the real field and complex field, respectively. We denote the set of all mth-order, n-dimensional real tensors by $\mathbb{R}^{[m,n]}$. For a tensor $\mathcal{A} \in \mathbb{R}^{[m,n]}$ and a vector $b \in \mathbb{R}^n$, a multilinear system is defined as

$$\mathcal{A}x^{m-1} = b, \tag{1.1}$$

where $x = [x_1, x_2, \ldots, x_n]^T$ is the unknown vector, and $\mathcal{A}x^{m-1}$ denotes the column vector whose ith entry is

$$(\mathcal{A}x^{m-1})_i = \sum_{i_2,\ldots,i_m=1}^{n} A_{ii_2\cdots i_m} x_{i_2} \cdots x_{i_m},$$

for $i = 1, 2, \ldots, n$. Multilinear systems of the form (1.1) arise in a number of applications, such as numerical partial differential equations, data mining, and tensor complementarity problems (see for example, [1–3]).

In their pioneering works, Qi [4] and Lim [5] independently introduced the concept of tensor eigenvalues. We say that $(\lambda, x) \in \mathbb{C} \times \mathbb{C}^n \setminus \{0\}$ is an eigenpair of a tensor $\mathcal{A} \in \mathbb{R}^{[m,n]}$ if

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]},$$

E-mail address: lxhan@umflint.edu.

http://dx.doi.org/10.1016/j.aml.2017.01.019

where $x^{[m-1]} = [x_1^{m-1}, x_2^{m-1}, \ldots, x_n^{m-1}]^T$. Let $\rho(\mathcal{A})$ denote the spectral radius of tensor $\mathcal{A}$, that is,

$$\rho(\mathcal{A}) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } \mathcal{A}\}.$$

Recently, the notion of M-tensors has been proposed and their properties have been studied in [6,7]. A tensor $\mathcal{A} \in \mathbb{R}^{[m,n]}$ is called an M-tensor if it can be written as $\mathcal{A} = s\mathcal{I} - B$, in which $\mathcal{I}$ is the mth-order, n-dimensional identity tensor, $\mathcal{B}$ is a nonnegative tensor (that is, each entry of $\mathcal{B}$ is nonnegative), and $s \geq \rho(\mathcal{B})$. Furthermore, $\mathcal{A}$ is called a nonsingular M-tensor if $s > \rho(\mathcal{B})$.

In [1], Ding and Wei investigated the solutions of the multilinear system (1.1) when the coefficient tensor $\mathcal{A}$ is an M-tensor. In particular, they show that the system (1.1) has a unique positive solution if $\mathcal{A}$ a nonsingular M-tensor and b is a positive vector (see [1, Theorem 3.2]). They generalized the Jacobi and Gauss–Seidel methods for linear systems to find the unique positive solution of the multilinear system (1.1). They also proposed to use Newton's method when the nonsingular M-tensor $\mathcal{A}$ is symmetric and numerically showed that Newton's method is much faster than the other methods. However, it is unclear whether or not the Newton method proposed in [1] always works when $\mathcal{A}$ is not symmetric.

In this paper, we propose a homotopy method for finding the unique positive solution of the multilinear system (1.1) and prove its convergence. The homotopy method is implemented using an Euler–Newton prediction–correction approach for path tracking. Numerical experiments show the efficiency of our method.

The paper is organized as follows. We introduce our homotopy method and prove its convergence in Section 2. Then we give some numerical results in Section 3.

2. A homotopy method

We are to design a homotopy method for finding the unique positive solution of the system (1.1) when $\mathcal{A}$ is a nonsingular M-tensor and b is a positive vector. For this purpose, we will solve the following polynomial system:

$$P(x) = \mathcal{A}x^{m-1} - b = 0. \tag{2.1}$$

We choose the starting system

$$Q(x) = \mathcal{I}x^{m-1} - b = 0, \tag{2.2}$$

and construct the following homotopy

$$H(x,t) = (1-t)Q(x) + tP(x) = 0, \quad t \in [0,1]. \tag{2.3}$$

Note that the starting system (2.2) trivially has a unique positive solution

$$x_0 = [b_1^{1/(m-1)}, b_2^{1/(m-1)}, \ldots, b_n^{1/(m-1)}]^T. \tag{2.4}$$

Moreover, the homotopy $H(x,t)$ can be expressed as

$$H(x,t) = (t\mathcal{A} + (1-t)\mathcal{I})x^{m-1} - b. \tag{2.5}$$

The partial derivatives matrix $D_xH(x,t)$ of the homotopy $H(x,t)$ plays an important role in our method. To compute this matrix, we need to partially symmetrize tensor $\mathcal{A} = (A_{i_1,i_2,\ldots,i_m})$ with respect to the indices $i_2, \ldots, i_m$. Specifically, we define the partially symmetrized tensor $\hat{\mathcal{A}} = (\hat{A}_{i_1,i_2,\ldots,i_m})$ by

$$\hat{A}_{i_1 i_2 \ldots i_m} = \frac{1}{(m-1)!}\sum_{\pi} A_{i_1 \pi(i_2 \ldots i_m)}, \tag{2.6}$$

where the sum is over all the permutations $\pi(i_2 \ldots i_m)$. The following lemma shows that this partial symmetrization preserves the nonsingular M-tensor structure.