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Multiplicity of nontrivial solutions for a critical degenerate Kirchhoff type problem[☆]

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ABSTRACT

In this paper, we study the following Kirchhoff type problem with critical growth

$$\begin{cases} -M\left(\int_\Omega |\nabla u|^2 \mathtt{d}x\right)\triangle u = \lambda |u|^2 u + |u|^4 u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^3$, $M \in C(\mathbb{R}^+, \mathbb{R})$ and $\lambda > 0$. We prove the existence of multiple nontrivial solutions for the above problem, when parameter λ belongs to some left neighborhood of the eigenvalue of the nonlinear operator $-M(\int_\Omega |\nabla u|^2 \mathtt{d}x)\triangle$.

1. Introduction

In this paper we are concerned with the multiplicity of nontrivial solutions for the following non-local problem:

$$\begin{cases} -M\left(\int_\Omega |\nabla u|^2 \mathtt{d}x\right)\triangle u = \lambda |u|^2 u + |u|^4 u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^3$. It is worthy noticing that the term $|u|^4 u$ reaches the Sobolev critical exponent since $2^* = 6$ in Dimension 3. We also observe that Problem (1.1) is related to the stationary analogue of the following equation

$$\begin{cases} u_{tt} - M\left(\int_\Omega |\nabla u|^2 \mathtt{d}x\right)\triangle u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

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Such kinds of equations as (1.2) fall outside the scope of the theory of classical elliptic equations because of their obvious nonlocal feature in $M\left(\int_\Omega |\nabla u|^2 \mathrm{d}x\right)$. Indeed, Eq. (1.2) are no longer a pointwise identity. The solvability of the Kirchhoff type equations for any dimension has already been extensively studied since Lions [1] introduced an abstracted framework to this problem, see for instance [2–4] and the references therein.

As we all know, there exist extensive literature devoted to the study of semilinear elliptic equations with critical growth after the celebrated paper by Brezis and Nirenberg [5]. In particular, people generalize those results in [5] to various kinds of elliptic equations either in bounded domain or the whole space (see [6–8] and the references therein). Recently, the study of Kirchhoff type equation with critical growth attracts lots of attention. For example, we refer to [9–12] for the problems in bounded domain, in which the variational techniques are developed when proving the existence of solutions. The main difficulty is due to the lack of compactness of the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^6(\Omega)$. In [9], the multiplicity of nontrivial solutions was obtained by using the genus theory. Combining with the method of Brezis and Nirenberg [5], Naimen [11] made use of cut-off techniques to derive the existence results. Based on [12], Yang, Liu and Ouyang also got the multiplicity of nontrivial solutions. For problems in the whole space, we refer to [13–16] and the references therein.

Note that problem (1.1) is said to be degenerate if $M(0) = 0$, and non-degenerate if $M(0) > 0$ (see [17]). We also point out that the above mentioned works only cover the non-degenerate case. To the best of our knowledge, there are very few results employing variational methods to consider the degenerate case of problem (1.1). We refer the readers to [17–22] and the references therein. Motivated by the above observations, our main goal in this paper is to investigate multiplicity of nontrivial solutions of the degenerate Kirchhoff problem (1.1). Our method is based on the concentration-compactness principle (see [23]) together with an abstract critical point theorem (see [7]), which relates to the $\mathbb{Z}_2$-cohomological index.

Hereafter, we need the following notations.

- For any $\rho > 0$ and for any $z \in \mathbb{R}^3$, $B_\rho(z)$ denotes the ball of radius ρ centered at z and $|B_\rho(z)|$ denotes its Lebesgue measure. Let $H := H_0^1(\Omega)$ be the Sobolev space equipped with the inner product and the norm $\langle u, v\rangle = \int_\Omega \nabla u \nabla v \mathrm{d}x$, $\|u\| := \langle u, u\rangle^{\frac{1}{2}}$, respectively. Let $|\cdot|_s$ be the usual L^s-norm and therefore the embedding $H \hookrightarrow L^s(\Omega)$ is continuous for $s \in [1, 2^*]$, and is compact for $s \in [1, 2^*)$, where $2^* = 6$. Denote by S the best Sobolev constant, where $S := \inf\{\|u\|^2/|u|_{2^*}^2 : u \in H_0^1(\Omega) \setminus \{0\}\}$.

Now we consider the Kirchhoff type nonlinear eigenvalue problem:

$$\begin{cases} -\left(\int_\Omega |\nabla u|^2 \mathrm{d}x\right) \triangle u = \lambda u^3 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

In [24], the authors used Morse theory to prove the existence of unbounded sequences of eigenvalues of the above problem. However, it does not seem to be known whether such a sequence coincides with the standard sequence of eigenvalues defined by using the genus. We recall some details as follows. Define

$$\Psi(u) = \frac{1}{\int_\Omega |u|^4 dx}, \quad u \in \mathcal{M} = \left\{u \in H_0^1(\Omega) : \left(\int_\Omega |\nabla u|^2 dx\right)^2 = 1\right\}. \tag{1.4}$$

Then eigenvalues of problem (1.3) on $\mathcal{M}$ coincide with critical values of Ψ. Set

$$\Psi^a = \{u \in \mathcal{M} : \Psi(u) \le a\}, \quad \Psi_a = \{u \in \mathcal{M} : \Psi(u) \ge a\}, \quad a \in \mathbb{R},$$

and $\mathcal{F}$ denotes the class of symmetric subsets of $\mathcal{M}$ and $i(\mathcal{M})$ stands for the $\mathbb{Z}_2$-cohomological index of $\mathcal{M} \in \mathcal{F}$ which will be introduced in Section 3. Set

$$\lambda_k := \inf_{M \in \mathcal{F}, i(M) \ge k} \sup_{u \in M} \Psi(u), \quad k \in \mathbb{N}.$$