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A note on oscillation of second-order delay differential equations

Jozef Džurina, Irena Jadlovská *

Department of Mathematics and Theoretical Informatics, Faculty of Electrical Engineering and Informatics, Technical University of Košice, B. Němcovej 32, 042 00 Košice, Slovakia

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ABSTRACT

The purpose of this paper is to study the second-order half-linear delay differential equation

$$\left(r(t)\left(y'(t)\right)^{\alpha}\right)' + q(t)y^{\alpha}(\tau(t)) = 0$$

under the condition $\int^{\infty} r^{-1/\alpha}(t)\mathrm{d}t < \infty$. Contrary to most existing results, oscillation of the studied equation is attained via only one condition. A particular example of Euler type equation is provided in order to illustrate the significance of our main results.

1. Introduction

In this paper, we are concerned with the oscillatory behavior of the second-order half-linear delay differential equation of the form

$$\left(r(t)(y'(t))^{\alpha}\right)' + q(t)y^{\alpha}(\tau(t)) = 0, \quad t \ge t_0, \tag{1.1}$$

where $\alpha > 0$ is a quotient of odd positive integers, r, $\tau \in C^1([t_0,\infty),(0,\infty))$ and $q \in C([t_0,\infty),[0,\infty))$. We also suppose that, for all $t \ge t_0$, $\tau(t) \le t$, $\tau'(t) \ge 0$, $\lim_{t\to\infty}\tau(t) = \infty$, and q does not vanish identically on any half-line of the form $[t_*,\infty)$.

Under the solution of Eq. (1.1) we mean a function $y \in \mathcal{C}([t_a,\infty),\mathbb{R})$ with $t_a = \tau(t_b)$, for some $t_b \ge t_0$, which has the property $r(y')^{\alpha} \in \mathcal{C}^1([t_a,\infty),\mathbb{R})$ and satisfies (1.1) on $[t_b,\infty)$. We consider only those solutions of (1.1) which exist on some half-line $[t_b,\infty)$ and satisfy the condition $\sup\{|x(t)| : t_c \le t < \infty\} > 0$ for any $t_c \ge t_b$. As is customary, a solution $y(t)$ of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is said to be nonoscillatory. The equation itself is termed oscillatory if all its solutions oscillate.

* Corresponding author.
E-mail addresses: jozef.dzurina@tuke.sk (J. Džurina), irena.jadlovska@tuke.sk (I. Jadlovská).

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Following Trench [1], we shall say that Eq. (1.1) is in canonical form if

$$R(t) := \int_{t_0}^{t} r^{-1/\alpha}(s)\mathrm{d}s \to \infty \quad \text{as} \quad t \to \infty. \tag{1.2}$$

Conversely, we say that (1.1) is in non-canonical form if

$$\pi(t) := \int_{t}^{\infty} r^{-1/\alpha}(s)\mathrm{d}s < \infty. \tag{1.3}$$

There is a significant difference in the structure of nonoscillatory (say positive) solutions between canonical and non-canonical equations. It is well known that the first derivative of any positive solution y of (1.1) is of one sign eventually, while (1.2) ensures that this solution is increasing eventually. Most often, Eq. (1.1) has been studied exactly in canonical form, see, e.g., [2–4] and the references therein.

On the other hand, when investigating non-canonical equations, both sign possibilities of the first derivative of any positive solution y have to be treated. A common approach in the literature (see [5–12]) for investigation of such equations consists in extending known results for canonical ones.

The objective of this paper is to study oscillatory and asymptotic properties of (1.1) in non-canonical form. Thus, in the sequel and without further mentioning, it will be always assumed that (1.3) holds.

In what follows, we briefly review several important oscillation results established for second-order non-canonical equations which can be seen as a motivation for this paper.

Theorem A *([11, Theorem 3.1]). Assume that*

$$\int^{\infty} \left(R^{\alpha}(\tau(t))q(t) - \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} \frac{\tau'(t)}{R(\tau(t))r^{1/\alpha}(\tau(t))} \right) \mathrm{d}t = \infty \tag{1.4}$$

and there exists a continuously differential function $\rho(t)$ such that $\rho(t) > 0$, $\rho'(t) \ge 0$ for $t \ge t_0$,

$$\int^{\infty} \rho(t)q(t)\mathrm{d}t = \infty \tag{1.5}$$

and

$$\int^{\infty} \left(\frac{1}{r(t)\rho(t)} \int^{t} \rho(s)q(s)\mathrm{d}s \right)^{1/\alpha} \mathrm{d}t = \infty. \tag{1.6}$$

Then every solution $y(t)$ of (1.1) oscillates or $\lim_{t\to\infty} y(t) = 0$.

Note that (1.4) in Theorem A eliminates existence of positive increasing solutions of (1.1), while conditions (1.5)–(1.6) ensure that any positive decreasing solution converges to zero in the neighborhood of infinity.

Recently, Mařík [9] revised Theorem A and provided its simplified version.

Theorem B *([9, Theorem 2]). With no lack of generality we can put $\rho \equiv 1$ in Theorem A and the pair of conditions (1.5) and (1.6) can be safely and with no lack of generality replaced by one condition*

$$\int^{\infty} \left(\frac{1}{r(t)} \int^{t} q(s)\mathrm{d}s \right)^{1/\alpha} \mathrm{d}t = \infty. \tag{1.7}$$

Erbe et al. [6,7], Hassan [8], and Saker [10] independently obtained the analogue of Theorem B in such sense that (1.4) has been replaced by another condition resulting from the use of different techniques.

In [6], authors also presented the following criterion which removes the possibility that any positive solution of (1.1) is decreasing.