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Constitutive-relation-error-based *a posteriori* error bounds for a class of elliptic variational inequalities

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ABSTRACT

On the basis of the dual variational formulation of a class of elliptic variational inequalities, a constitutive relation error is defined for the variational inequalities as an *a posteriori* error estimator, which is shown to guarantee strict upper bounds of the global energy-norm errors of kinematically admissible solutions. A numerical example is presented to validate the strictly bounding property of the constitutive relation error for the variational inequalities in question.

1. Introduction

A class of nonlinear problems in science and engineering can be expressed in terms of variational inequalities [1]. This work is focused on some variational inequalities represented as quadratic minimizing problems in convex sets, which can be used to describe one-sided contact of elastic bodies in structural analysis and some free boundary problems in hydrodynamics, see [2,3]. When solving the variational inequalities numerically, *a posteriori* error estimation is often performed to evaluate the discretization error in the approximate solution, and some typical techniques are reported in [4–10] for estimating global norm errors.

The constitutive relation error [11,12] was proposed by Ladevèze in [13] as an *a posteriori* error estimator for finite element analysis. It has been applied to a variety of problems in mechanics and claimed to provide strict upper bounds for global errors [14,15] and strict upper & lower bounds for errors in quantities of interest [16,17]. In this paper, the constitutive relation error theory is extended to the class of elliptic variational inequalities in question. The constitutive relation error is established via the dual variational

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formulation of the variational inequalities, and its strictly upper-bounding property for global errors is shown to be inherited.

After the introduction, the elliptic variational inequality problem is defined in Section 2, and the corresponding constitutive relation error is defined in Section 3. In Section 4, some numerical techniques are developed in detail and the strictly bounding property of the constitutive relation error is illustrated by an example. Some conclusions are drawn in Section 5.

2. Problem definition

Consider the problems described by elliptic variational inequalities in the following (primal) quadratic minimizing formulation:

$$u = \arg\min_{v\in K} \left\{ \frac{1}{2}\langle \mathcal{A}v, \mathcal{A}v\rangle - l(v) + j(v) \right\}, \tag{1}$$

where $K \subset V$ is a closed convex nonempty set in a real Hilbert space V, $\langle\cdot,\cdot\rangle$ is the inner product of a Hilbert space S, $\mathcal{A} : V \to S$ is a linear differential operator, $l(\cdot)$ is a continuous linear form on V, i.e. $l \in V'$, and $j(\cdot)$ is a lower semi-continuous convex but not necessarily differentiable functional defined on V. Define a bilinear form $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ as $a(u, v) = \langle \mathcal{A}u, \mathcal{A}v\rangle$, $(u, v) \in V \times V$, and assume that a is continuous and coercive. Then the variational formulation (1) can be rewritten as

$$u = \arg\min_{v\in K} \left\{ \frac{1}{2} a(v, v) - l(v) + j(v) \right\}, \tag{2}$$

or equivalently: $u \in K$ such that

$$a(u, v - u) + j(v) - j(u) \geq l(v - u) \quad \forall v \in K. \tag{3}$$

Suppose that the convex set K can be defined in terms of a convex cone M, which is a subset of a real Hilbert space L and has its vertex at θ_L (zero element of L), and a linear form $g_1 \in L'$, i.e.

$$K = \{v \in V : b_1(v, \eta) \geq g_1(\eta), \forall \eta \in M\}, \tag{4}$$

with $b_1(\cdot,\cdot)$ being a continuous bilinear form on $V \times L$. Assume that functional j is defined in terms of a bounded subset N of a real Hilbert space Q, such that

$$j(v) = \max_{\xi\in N} \{-b_2(v, \xi) + g_2(\xi)\}, \quad v \in V, \tag{5}$$

in which $b_2(\cdot,\cdot)$ is a continuous bilinear form on $V \times Q$ and $g_2 \in Q'$. Suppose that bilinear forms b_1 and b_2 satisfy the following inf–sup condition: there exists a constant $\beta > 0$ such that

$$\sup_{v\in V\setminus\{\theta_V\}} \frac{|b_1(v, \eta) + b_2(v, \xi)|}{\|v\|_V} \geq \beta \left(\|\eta\|_L^2 + \|\xi\|_Q^2 \right)^{\frac{1}{2}} \quad \forall (\eta, \xi) \in L \times Q, \tag{6}$$

where $\|\cdot\|_V$, $\|\cdot\|_L$ and $\|\cdot\|_Q$ denote the norms on spaces V, L and Q, respectively.

With the definition of the set K, i.e. an inequality constraint in (4), and that of the functional j in (5), a Lagrangian formulation is defined by introducing two multipliers as:

$$\mathcal{L}(v, \eta, \xi) = \frac{1}{2} a(v, v) - l(v) - b_1(v, \eta) + g_1(\eta) - b_2(v, \xi) + g_2(\xi). \tag{7}$$

There exists only one saddle point of $\mathcal{L}$ in $V \times M \times N$, denoted by (u, λ, ω), such that

$$\mathcal{L}(u, \eta, \xi) \leq \mathcal{L}(u, \lambda, \omega) \leq \mathcal{L}(v, \lambda, \omega) \quad \forall (v, \eta, \xi) \in V \times M \times N, \tag{8}$$