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Models of space-fractional diffusion: A critical review

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ABSTRACT

Space-fractional diffusion problems are investigated from the modeling point of view. It is pointed out that the elementwise power of the Laplacian operator in $\mathbb{R}^n$ is an inadequate model of fractional diffusion. Also, the approach with fractional calculus using zero extension is not a proper model of homogeneous Dirichlet boundary conditions. At the time, the spectral definition of the fractional Dirichlet Laplacian seems to be in many aspects a proper model of fractional diffusion.

1. Introduction

The accurate measurement techniques in the last decades confirmed the occurrence of the space-fractional (or anomalous) diffusion in a wide range of real-life phenomena. By tracking individual particles, it is possible to estimate their average displacement $\langle|s(t)|\rangle$ over a short time interval $(0,t)$. While in a standard diffusion process the linear dependence $\langle|s(t)|\rangle \sim t^{\frac{1}{2}}$ is valid, in many cases, the proportionality $\langle|s(t)|\rangle \sim t^{\frac{\alpha}{2}}$ or $\langle|s(t)|^2\rangle \sim t^{\alpha}$ can be detected with $\alpha \neq 1$. In a probabilistic interpretation, the random displacement represents an α-stable Lévy process [1] so that $s(t)$ is distributed with some t-dependent density.

Also, a number of continuous deterministic models have been proposed, where the non-local spatial operators were associated with the anomalous diffusion. Based on these, various numerical methods were developed started with the works in [2] and [3]. A critical point of all PDE models is to incorporate and use boundary conditions. From the point of view of the functional analysis, they are already necessary to define the differential operators. In many real situations, one should use Neumann boundary conditions in the models. A corresponding analysis can be found in [4] and [5].

In the literature, mostly problems with homogeneous Dirichlet boundary conditions were investigated; a systematic study of incorporating inhomogeneous data has just been started [6]. The aim of this contribution

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is to give an overview of these approaches from the point of view how realistic models they deliver for the anomalous diffusion. Therefore, we focus on the multidimensional models.

2. Mathematical preliminaries: basic models for anomalous diffusion

Differential operators corresponding to the diffusion will be defined on a Lipschitz domain $\Omega \subset \mathbb{R}^d$. They are all non-local in the sense that the flux at a given point $\mathbf{x}$ depends on the density function in a neighborhood of $\mathbf{x}$. For $\Omega = \mathbb{R}^d$, they can be defined on the Schwartz space of rapidly decreasing functions. Usually, we do not give explicitly the largest linear space where these operators are defined. In each case, the positive constant $\frac{\alpha}{2}$ denotes the exponent of the classical diffusion operator, where in all cases $\alpha \in (0, 2]$.

For a bounded interval $(a, b) = \Omega$, the most popular choice is the symmetric Riemann–Liouville derivative $\partial^\alpha_{\mathrm{RL}}$, which is defined for $1 < \alpha < 2$ with

$$\partial^\alpha_{\mathrm{RL}} u(x) = \frac{1}{2\Gamma(2-\alpha)} \partial^2 \left\{ x \to \int_a^x u(s)(x-s)^{1-\alpha}\,\mathrm{d}s + \int_x^b u(s)(s-x)^{1-\alpha}\,\mathrm{d}s \right\}. \tag{1}$$

Another frequently used definition (for $1 < \alpha < 2$ and $u \in C^2[a, b]$) was suggested by Caputo:

$$\partial^\alpha_{\mathrm{C}} u(x) = \frac{1}{2\Gamma(2-\alpha)} \int_a^x u''(s)(x-s)^{1-\alpha}\,\mathrm{d}s + \int_x^b u''(s)(s-x)^{1-\alpha}\,\mathrm{d}s. \tag{2}$$

This definition can be used, *e.g.*, directly in a model of neuronal transmission [7]. For the relation of (1) and (2), we refer to [8], Theorem 2.7.

Based on the classical theory of fractional order derivatives [9], a number of alternative definitions are available, which are, in general, not equivalent. For a clear overview on these, we refer to [8] and [10].

For $\Omega = \mathbb{R}$, many of the different definitions coincide [11]. We mention three of them, which are linked to the operators in this section. The spectral definition of the fractional Laplacian reads as

$$(-\Delta)^{\frac{\alpha}{2}} u = \mathcal{F}^{-1}(|\mathtt{Id}|^\alpha \mathcal{F} u), \tag{3}$$

where $\mathcal{F}$ denotes the Fourier transform and $\mathtt{Id}$ the identity function. The following definition was proposed by M. Riesz

$$\partial^\alpha_{\mathrm{R}} u(x) = -\frac{1}{2\Gamma(2-\alpha)\cos(\alpha\pi/2)} \partial^2 \left\{ x \to \int_{-\infty}^x u(s)(x-s)^{1-\alpha}\,\mathrm{d}s + \int_x^\infty u(s)(s-x)^{1-\alpha}\,\mathrm{d}s \right\}, \tag{4}$$

which is a generalization of (1). For its definition domain, we refer to [5].

As a third one, we mention Balakrishnan's definition:

$$(-\Delta)^{\frac{\alpha}{2}} u = \frac{\sin \alpha\pi/2}{\pi} \int_0^\infty \Delta(s \cdot \mathtt{Id} - \Delta)^{-1} u \cdot s^{\alpha/2-1}\,\mathrm{d}s,$$

which applied for the sin function with the well-known identity $\int_0^\infty \frac{s^{\alpha/2-1}}{s+1}\,\mathrm{d}s = \frac{\pi}{\sin(\alpha\pi/2)}$ (see, [12], 3.222) gives

$$(-\Delta)^{\frac{\alpha}{2}} \sin = \frac{\sin(\alpha\pi/2)}{\pi} \int_0^\infty \sin \cdot \frac{1}{s+1} \cdot s^{\alpha/2-1}\,\mathrm{d}s = \sin.$$

Therefore, we have the following identities for the trigonometric functions:

$$(-\Delta)^{\frac{\alpha}{2}} \sin = \partial^\alpha_{\mathrm{R}} \sin = \sin \qquad \text{and} \qquad (-\Delta)^{\frac{\alpha}{2}} \cos = \partial^\alpha_{\mathrm{R}} \cos = \cos. \tag{5}$$

For $\Omega = \mathbb{R}^d$, the spectral definition in (3) can be applied without any change.