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Inviscid traveling waves of monostable nonlinearity

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A B S T R A C T

Inviscid traveling waves are ghost-like phenomena that do not appear in reality because of their instability. However, they are the reason for the complexity of the traveling wave theory of reaction–diffusion equations and understanding them will help to resolve related puzzles. In this article, we obtain the existence, the uniqueness and the regularity of inviscid traveling waves under a general monostable nonlinearity that includes non-Lipschitz continuous reaction terms. Solution structures are obtained such as the thickness of the tail and the free boundaries.

1. Phantom of traveling waves

Traveling wave solutions with a monostable nonlinearity have been intensively studied (see [1]). For example, consider a reaction diffusion equation,

$$u_t = d(u^m)_{xx} + u^\beta(1 - u^\alpha), \quad t, \alpha, \beta, d, m > 0,\ x \in \mathbf{R}, \tag{1.1}$$

where subindexes indicate partial derivatives. The usual traveling wave phenomenon is produced by a correlation between diffusion and reaction. However, there are phantom-like traveling waves for any speed $c \in \mathbf{R}$ which are produced entirely by reaction ($d = 0$). The reason why the reaction–diffusion equation admits a traveling wave of any speed greater than a minimum one, $|c| \geq c^* > 0$, is related to such traveling waves.

These phantom-like traveling wave solutions satisfy an inviscid equation,

$$v_t = v^\beta(1 - v^\alpha), \quad t > 0,\ x \in \mathbf{R}, \tag{1.2}$$

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where $v(x,\cdot)$ solves the ODE independently for each $x \in \mathbf{R}$. Consider a traveling wave solution of speed $c > 0$, $v(x,t) = v(x-ct)$ (here, we are abusing notation by using the same "v" for the traveling wave profile). Then, $v = v(z)$ satisfies

$$cv' + v^{\beta}(1 - v^{\alpha}) = 0, \qquad \alpha, \beta > 0, z \in \mathbf{R}. \tag{1.3}$$

We restrict our study to a traveling wave with monotonicity. The solution is global and unique at least for $\beta \geq 1$ by the Cauchy Lipschitz theorem and satisfies boundary conditions

$$\lim_{z\to-\infty} v(z) = 1, \quad \lim_{z\to\infty} v(z) = 0, \quad v(0) = 0.5. \tag{1.4}$$

Here, we have chosen a decreasing traveling wave. Since a traveling wave is invariant in translation, the extra condition $v(0) = 0.5$ is taken for the uniqueness. An inviscid traveling wave, denoted by $v = v_{c,\alpha,\beta}$, depends on three parameters, c, α, β.

For the Fisher equation case ($\alpha = \beta = 1$) the inviscid traveling wave is simply the logistic function given in (2.3). This information of inviscid traveling waves was the key to obtain the connection between viscous and inviscid traveling waves (see [2]). The purpose of this paper is to obtain the properties of inviscid traveling waves required to show similar connections in the general setting of the above.

2. Three examples of inviscid traveling waves

We consider three cases of inviscid traveling waves solutions. They may have an algebraic tail, an exponential tail, or a free boundary, respectively.

Case 1. $\alpha = 1, \beta = 2$ (algebraic tail). Separate variables in (1.3) and obtain $-\frac{c}{v^2(1-v)}v' = 1$. Integrate both sides on $(0, z)$ and obtain

$$-\int_0^z \frac{cv'(s)}{v^2(s)(1-v(s))} ds = z.$$

A change of variable and the condition $v(0) = 1/2$ yield that

$$-\int_0^z \frac{cv'(s)}{v^2(s)(1-v(s))} ds = -\int_{v(0)}^{v(z)} \frac{c}{v^2(1-v)} dv = c\left(\frac{1}{v(z)} + \log\frac{1-v(z)}{v(z)} - 2\right).$$

Therefore, we have

$$c\left(\frac{1}{v(z)} + \log\frac{1-v(z)}{v(z)} - 2\right) = z. \tag{2.1}$$

From this implicit formula, one may easily check the boundary conditions (1.4) and, furthermore,

$$\lim_{z\to\infty} z v_{c,\alpha=1,\beta=2}(z) = c. \tag{2.2}$$

Thus, the traveling wave has an algebraic tail $v_{c,\alpha=1,\beta=2}(z) \cong cz^{-1}$ for z large.

Case 2. $\alpha = 1, \beta = 1$ (exponential tail). In this case Eq. (1.3) is written as

$$v' = -\frac{1}{c}v(1-v), \quad z \in \mathbf{R}.$$

The traveling wave is the logistic function and is given by

$$v_{c,\alpha=1,\beta=1}(z) = \big(1 + \exp(z/c)\big)^{-1}. \tag{2.3}$$

This solution satisfies the conditions in (1.4) and $v_{c,\alpha=1,\beta=1}(z) \cong e^{-\frac{1}{c}z}$ for z large.