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Nonsmooth and discontinuous speed-gradient algorithms☆

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ABSTRACT

In this article, nonsmooth extensions of the Speed-Gradient (SG) algorithms in differential and finite forms are proposed. The conditions ensuring achievement of the control goal (convergence of the goal function to zero) are established. Furthermore, conditions under which the control goal is achieved in finite time with the use of nonsmooth or discontinuous SG algorithms are obtained. Theoretical results are illustrated by example of nonsmooth energy-based control for a non-affine in control pendulum-like system.

1. Introduction

A number of approaches to stabilization of nonlinear systems are based on the introduction of an appropriate goal/objective functional and the design of a control algorithm providing its asymptotic optimization. Perhaps, most popular ones are based on the design of a controller that minimizes integral performance index. In many cases these approaches lead to a stable closed loop system [1–3]. Alternatively, in [4] it was proposed to minimize the rate of change of a function of the state $V(x)$ along trajectories of the controlled system at each time instant t. However, such local minimization is not necessary to achieve stability. If the function $V(x)$ serves as a Lyapunov function for the closed loop system, stability is ensured, provided $V(x(t))$ decreases for all t. Such class of algorithms have been studied since late 1970s under the name of Speed-Gradient algorithms [5,6] or (in the affine case) LgV or Jurdjevic–Quinn algorithms [7]. It was shown [8] that a variety of adaptation and control algorithms can be obtained with a proper choice of the controlled system or the goal function. A fundamental result was obtained by E.Sontag [9] who has demonstrated that under special choice of a scalar gain the closed loop becomes globally stable ('universal Sontag's construction').

Stability analysis for an overwhelming majority of the existing algorithms was performed under the assumption of smoothness of the goal function and continuity of the control system right hand sides. In such a case the function $\dot{V}$ is continuous. However, a relaxation of the smoothness assumption provides hopes for better performance of the closed loop system. Therefore it is important to develop a systematic theory allowing one to design nonlinear controllers and to prove stability in nonsmooth situations. Some special cases of relay algorithms were considered in [8]; they provided a new view of the variable structure systems (VSS). However, a comprehensive theory based on the well developed apparatus of nonsmooth analysis [10–14] did not exist until recently.

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The first attempt to study nonsmooth counterparts of SG-controllers was made in [15], where proofs were just outlined. In this paper, we provide a more detailed exposition and some extension of the results of [15]. Besides, a new interesting example is considered. Let us also mention the paper [16], in which a different version of a nonsmooth SG-algorithm was considered and utilized to design an almost global stabilizer of the Brockett integrator that is continuous along solutions of the closed-loop system.

As a basis for formal approach to nonsmooth problems we chose the concept of Hadamard directional differentiability [11]. It allowed us to introduce and analyse the Speed-Subgradient algorithms in differential and finite forms.

An important advantage of nonsmooth optimization and control algorithms is a potential for finite-time convergence: the control goal can be achieved in finite time. It is widely used in variable structure systems (VSS), in switching systems, etc., see [17–21] and references therein. In this paper, we show that under some additional assumptions nonsmooth and discontinuous SG-algorithms in finite form provide finite-time convergence to the goal set.

It is worth noting that in the smooth case the Speed-Gradient algorithms are defined via the gradient of the speed of change (i.e. Lie derivative) of a chosen goal function. In the nonsmooth case, one has to replace the gradient and the derivative with their nonsmooth counterparts, i.e. with a subdifferential and a generalized derivative. Being inspired by the ideas of [11], we utilized Hadamard's directional derivative and the subdifferential (in the sense of convex analysis) of an upper convex approximation of this derivative. Let us note that one can use different tools from nonsmooth and variational analysis, such as the subderivative and the proximal subdifferential [13], in order to construct a different extension of the Speed-Gradient algorithms to a nonsmooth setting.

The structure of the paper is as follows. In Section 2 necessary notions and methods from nonsmooth analysis are outlined. In Section 3 the problem statement is given and two key results concerning nonsmooth SG-algorithms in differential and finite forms are presented. Two examples are described in Section 4.

2. Preliminaries

In this section, we recall some notions from nonsmooth analysis [11] and set-valued analysis [22] that are used throughout the article. Denote by $|\cdot|$ the Euclidean norm in $\mathbb{R}^n$, and denote $\mathbb{R}_+ = [0, +\infty)$.

Let a real-valued function f be defined in a neighbourhood of a point $x \in \mathbb{R}^n$. The function f is called *Hadamard directionally differentiable* at the point x if for any $v \in \mathbb{R}^n$ there exists the finite limit

$$f'(x; v) = \lim_{[\alpha, v'] \to [+0, v]} \frac{f(x + \alpha v') - f(x)}{\alpha}$$

(the motivation behind the notation under lim was discussed in [23]). The function $f'(x; \cdot)$ is called the *Hadamard directional derivative* of f at x. Note that there exists the elaborate calculus of Hadamard directional derivatives [11]. Observe also that if $n = 1$, then the quantity $f'(x, 1)$ coincides with the right-hand side derivative of f at x that is denoted by $D_+ f(x)$.

It is easy to see that the function $v \to f'(x; v)$ is continuous and positively homogeneous (of degree one), i.e. for any $v \in \mathbb{R}^n$ and $\lambda \geq 0$ one has $f'(x; \lambda v) = \lambda f'(x; v)$. A convex positively homogeneous function $p : \mathbb{R}^n \to \mathbb{R}$ such that $p(v) \geq f'(x; v)$ for all $v \in \mathbb{R}^n$ is called an *upper convex approximation* of the function $f'(x; \cdot)$.

Let $C \subset \mathbb{R}^n$ be an open set. Recall that a function F that maps points from C to possibly empty subset of $\mathbb{R}^m$ is called a *set-valued mapping* (or *multifunction*) from C to $\mathbb{R}^m$. The set-valued mapping F is called *outer semicontinuous* at a point $x_0 \in C$ if for any open set $V \subset \mathbb{R}^m$ with $F(x_0) \subset V$ there exists $\delta > 0$ such that for all $x \in C$ with $|x - x_0| < \delta$ one has $F(x) \subset V$. The set-valued mapping F is called *measurable* if for any open set $V \subset \mathbb{R}^n$ the set $\{x \in C : F(x) \cap V \neq \emptyset\}$ is measurable. One can show that any outer semicontinuous set-valued mapping is measurable.

An important example of an outer semicontinuous (and thus measurable) set-valued mapping is the subdifferential mapping of a convex function. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function. It should be noted that throughout this article we consider only finite-valued convex functions.

Recall that the set

$$\partial f(x) = \left\{ v \in \mathbb{R}^n : f(y) - f(x) \geq v^T (y - x) \quad \forall y \in \mathbb{R}^n \right\}$$

is referred to as the *subdifferential* of f at a point x. One can verify that the set $\partial f(x)$ is nonempty, convex and compact. As it was mentioned above, the subdifferential mapping $x \to \partial f(x)$ is outer semicontinuous on $\mathbb{R}^n$.

3. Nonsmooth speed-gradient: Two algorithms

3.1. Problem formulation

Consider the controlled system

$$\dot{x} = F(x, u, t), \quad t \geq 0, \tag{1}$$

where $x \in \mathbb{R}^n$ is the vector of the system state, and $u \in \mathbb{R}^m$ is the control. We assume that the function $F : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}_+ \to \mathbb{R}^n$ satisfies the Carathéodory condition, i.e. the mapping $(u, x) \to F(x, u, t)$ is continuous for almost all $t \geq 0$, and the mapping $t \to F(x, u, t)$ is measurable for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$. Unless otherwise stated, a solution of (1), even in the case of a