ARTICLE IN PRESS

Nonlinear Analysis: Hybrid Systems ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Nonlinear Analysis: Hybrid Systems

journal homepage: www.elsevier.com/locate/nahs

Causal modeling in impulsive systems: A new rigorous non-standard analysis approach

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ARTICLE INFO

Article history:
Available online xxxx

Keywords:
Impulsive system
Impulsive control
Nonstandard analysis
Generalized solution
Systems with impacts

ABSTRACT

A new approach for modeling nonlinear impulsive system is suggested based on nonstandard analysis. Basic properties of the hyperreals in nonstandard analysis are revisited. Depending on the convergence rate of infinitesimals in hyperreals, a new extended real space is proposed, which extends the one dimensional real line to a countably infinite dimensional extended real space. Generalized functions are defined via a sequential approach on the extended space, which yields a class of Heaviside functions and singular functions. By using the extended functions, a causal way for characterizing jumps in discontinuous system follows. We illustrate the usefulness of the theoretical development by analyzing three simple cases of impulsive affine system: (1) scalar case, (2) multi-dimensional case, and (3) one dimensional horizontal bouncing ball. The results suggest not only the existence of such infinitesimal models within the jump but also how to detour the equilibrium point while connecting the discontinuous state at the impact.

1. Introduction

Impulsive systems form a subclass of hybrid systems where the dynamics of motions are modeled with a continuous vector field in the presence of jumps. Earlier work on impulsive differential equations and their dynamics can be found in [1–3]. These works have successfully built a theory with formal definitions and initial assumptions on the impulsive systems. In addition, fundamental analysis on the system properties of the impulsive systems was thoroughly analyzed in the book, [4]. Beyond the theory, it extends to the applications in biological system and ecological system in [5–7]. Furthermore, a numerical approach on solving ODEs with discontinuous vector fields can be found in [8,9].

By using the classical definition in [1], the impulsive system is an interaction between the continuous dynamics and the reset maps where the timing of the reset can be explicitly known or implicitly obtained by additional state dependent equations. In either case, the reset map should fully describe how the jump will occur at the moment of the reset or an impact. Suppose that $\{t_i\}_i$ is a monotonically increasing sequence of reset times where $\lim_{i\to\infty} t_i = \infty$, then the formulation of the nth order impulsive equation is given as

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)) \quad \text{if } t \in \mathcal{T} \setminus \bigcup_{i=0}^{\infty} t_i \\ \Delta x(t_i) = k(x(t_i-)) \quad \text{if } t \in \bigcup_{i=0}^{\infty} t_i, \end{cases}, \tag{1}$$

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http://dx.doi.org/10.1016/j.nahs.2017.01.007
1751-570X/

Please cite this article in press as: N.-s.P. Hyun, E.I. Verriest, Causal modeling in impulsive systems: A new rigorous non-standard analysis approach, Nonlinear Analysis: Hybrid Systems (2017), http://dx.doi.org/10.1016/j.nahs.2017.01.007

Nomenclature

$\langle r; s\rangle$	A geometric sequence: $\{s \cdot r^n\}_n$ for $s, r \in \mathbb{R}$
$\mathbb{R}^{\mathbb{R}}$	The set of real valued functions on domain $\mathbb{R}$
$\{a_n\}_n$	A real valued sequence: $a_n \in \mathbb{R}$ for all $n \in \mathbb{N}$
$\{a_n\}_{n=1}^N$	A set of elements in $\mathbb{R}$: $a_n \in \mathbb{R}$ for all $n \in \{1, \ldots, N\}$
$\{f_n\}_n$	A sequence of functions: $f_n \in \mathbb{R}^{\mathbb{R}}$ for all $n \in \mathbb{N}$
$C(\mathbb{R})$	The set of real valued continuous functions
$C^1(\mathbb{R})$	The set of real valued continuously differentiable functions
$D^1(\mathbb{R})$	The set of real valued right differentiable functions
$PC(\mathbb{R})$	The set of real valued piecewise continuous functions

where $\mathcal{T} = [0, \infty)$ is the time of evolution. Suppose that $u : \mathcal{T} \to \mathbb{R}$ is a piecewise continuous function, and $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is a nth order vector function such that $f(\cdot, t)$ is Lipschitz and $f(x, \cdot)$ is piecewise continuous on t, and $k : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function, then given an initial condition, $x_0 \in \mathbb{R}^n$, there exists a unique solution to Eq. (1). The instantaneous change of the state is predefined by the jump equation in Eq. (1) where $\Delta x(t_i)$ is the state jump, $x(t_i+) - x(t_i-)$, and $x(t_i+) := \lim_{t \overset{>}{\to} t_i} x(t)$ and $x(t_i-) := \lim_{t \overset{<}{\to} t_i} x(t)$ are the right and left limit points at t_i. We call this classical formulation as an *impulsive effect system* or *effect model* since the dynamics are described without modeling the *cause* of the instantaneous changes.

However, there are many applications, especially, in mechanical systems with collision of rigid bodies where modeling the cause of the reset or controlling the impact is important. Although the majority of the research on generating the walking gait of a bipedal robot relies on the impulsive effect system, it is crucial to verify whether the reset in Eq. (1) is compatible with the force and torque components of the impacts unless the assumed impulsive effect model is invalid. See Section 4.6 in [10] for detailed explanation of impact model in walking robots. Recently, a contact force model in a relatively short duration of quadrupedal robot on a ground has been modeled and tested for the MIT Cheetah robot in [11]. By controlling the contact force, the authors in [11] were able to make the robot to run at a desired speed without tuning the reset mapping or any other control parameters. In addition, numerous nonlinear models of the contact force during the contact of two bodies were studied in [12]. Beyond the application to the walking robots, more examples of impact modeling can be found in [13] for controlling an air hockey puck, and in [14] for controlling the ball with a racket with different stiffness, and references therein.

1.1. Causal modeling of impulsive systems

In this paper, we consider a general mathematical problem of modeling the cause of the impulsive effect system. Since the change of the state at the impact occurs in an instantaneous fashion, we aim to find the equivalent generalized ordinary differential equation (GODE) driven by a singular function, which generates the same solution to a given impulsive effect equation in Eq. (1), if it exists. We call such an equivalent GODE a *causal model* or a *causal representation* of a given impulsive effect system.

Here is a motivating example. Given a simple autonomous linear impulsive system with constant jumps,

$$\begin{cases} \dot{x}(t) = Ax(t) & \text{if } t \in \mathcal{T} \setminus \bigcup_{i=0}^{\infty} t_i \\ \Delta x(t_i) = c_i & \text{if } t \in \bigcup_{i=0}^{\infty} t_i, \end{cases} \tag{2}$$

where $A \in \mathbb{R}^{n\times n}$, and $c_i \in \mathbb{R}^n$ corresponds to the constant jump at each t_i. Suppose that there exists a matrix $B \in \mathbb{R}^n$ such that (A, B) is completely reachable, then there exists a singular function $u_s = \sum_{k=1}^{\infty} \sum_{i=0}^{n-1} u_{i,k}\delta^{(i)}(t - t_k)$ where $\delta^{(i)}$ is the ith distributional derivative of delta function and $\{u_{i,k}\}_{k=0}^{n-1} \subset \mathbb{R}$ is a function of A, B and c_i for each $i \in \mathbb{N}$, such that the trajectory of the GODE,

$$\dot{x}(t) = Ax(t) + Bu_s, \tag{3}$$

coincides with the solution to the impulsive effect equation, Eq. (2). The existence of such singular control is given by the instantaneous reachability condition in linear system theory. See [15] for detailed derivation of $\{u_{i,k}\}_{k=0}^{n-1}$.

Finding the causal representation of this particular example is feasible because of two conditions: (1) the instantaneous state jump, $\Delta(x(t_i))$ for all $i \in \mathbb{N}$, was independent from the previous state $x(t_i-)$, and (2) we propose that the instantaneous jump can be modeled by simply multiplying a constant matrix B to the singular function. By satisfying these conditions, the state can be successively differentiated, and higher derivatives of the singular function are not multiplied with the state dependent matrix.