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# Computation of periodic solutions in maximal monotone dynamical systems with guaranteed consistency

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ABSTRACT

In this paper, we study a class of set-valued dynamical systems that satisfy maximal monotonicity properties. This class includes linear relay systems, linear complementarity systems, and linear mechanical systems with dry friction under some conditions. We discuss two numerical schemes based on time-stepping methods for the computation of the periodic solutions when these systems are periodically excited. We provide formal mathematical justifications for the numerical schemes in the sense of consistency, which means that the continuous-time interpolations of the numerical solutions converge to the continuous-time periodic solution when the discretization step vanishes. The two time-stepping methods are applied for the computation of the periodic solution exhibited by a power electronic converter and the corresponding methods are compared in terms of approximation accuracy and computation time.



## 1. Introduction

Set-valued dynamical systems and differential inclusions play an important role in many branches of science and engineering [1]. An important concept in this context is the maximal monotonicity of the involved set-valued mappings. There is a large body of literature on the use of maximal monotonicity in mathematics [2–4], and in recent years this property was also exploited in the context of non-smooth dynamical systems and hybrid systems such as linear complementarity systems [5–10], linear relay systems [11,12], piecewise linear systems, projected dynamical systems [13–15], and applications including electrical networks with switching elements as in power converters [7,10,16–18], constrained mechanical systems [19,20], and systems with dry friction. In fact, in most of the above mentioned works non-smooth systems are perceived as the interconnection of a linear time-invariant (LTI) system and a static relationship described by a set-valued mapping, which turned out to be fruitful for the analysis. This perspective finds its origin in Lur'e systems, see, e.g., [21].

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In this paper, we also focus on non-smooth dynamical systems that arise from the interconnection of LTI systems and static set-valued mappings, although we will embed these systems in a general class of differential inclusions (DIs) that satisfy maximal monotonicity properties. The latter embedding has been used also in, e.g., [8,22,23], in which an essential assumption was the (strict) passivity of the LTI systems and the maximal monotonicity of the set-valued mapping, which imply that the resulting DIs indeed have maximal monotone right-hand sides. For this type of set-valued dynamical systems we consider the problem of numerical construction of periodic solutions when these systems are being periodically excited. Although many methods exist for numerical (forward) simulation of set-valued systems, see, e.g., the survey paper [24], numerical methods for constructing periodic solutions *including formal justifications* are limited. Therefore, we propose two numerical methods here for which such formal justifications can be given.

The first numerical method is based on time-discretization (time-stepping) [25–28] in combination with extensive simulation. This method relies on the asymptotic stability property of the searched periodic solution of the continuous-time system (and the asymptotic stability of the periodic solution of the discretized system) to warrant that sufficiently long numerical simulation recovers the periodic solution accurately. In fact, the property that we exploit is closely related to concepts such as incremental stability [29] (based on quadratic Lyapunov functions) and quadratic convergence [30,31]. As such, the work in this paper connects to results on (quadratic) convergence in the context of maximal monotone DIs such as, e.g., [32,33], and on incremental stability such as, e.g., [22] (although this terminology was not explicitly used in [22]). In fact, the concepts of quadratic convergence and Lyapunov-based characterizations for incremental stability can even be seen as a kind of maximal monotonicity properties in some situations (cf. Remark 1).

The second numerical method studied in this paper combines time-stepping techniques with two-point boundary value problems (to enforce periodicity), as used in, e.g., [34].

Both these classes of methods seem to work well in practice, but they often lack formal justification. Instead, in this paper, the numerical schemes are accompanied with a guarantee of *consistency*, in the sense that the 'exact' periodic solution (i.e., the one belonging to the continuous-time system) is recovered when the discretization period (and simulation window) tends to zero (and infinity, respectively). To the best of our knowledge, such proofs are not available in the literature. Building upon our preliminary work [35], in which no formal proofs were provided and more stringent conditions were needed, in this paper we do present *formal conditions under which the consistency can be guaranteed for both these methods (including rigorous proofs)* when applied to the class of non-smooth dynamical systems under study. Once the theoretical justification in the form of consistency is in place, we also provide a numerical example to illustrate the efficiency of the two methods and compare them in terms of approximation accuracy and required computation time.

The following notation will be used in the sequel. Closures and interiors of sets are denoted by cl and int. For a set-valued mapping $\mathcal{P} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ we denote the domain of $\mathcal{P}$, i.e. $\{x \in \mathbb{R}^n \mid \mathcal{P}(x) \neq \emptyset\}$, by dom $\mathcal{P}$. The graph gr($\mathcal{P}$) of $\mathcal{P}$ is given by $\{(x, x^*) \in \mathbb{R}^n \times \mathbb{R}^n \mid x^* \in \mathcal{P}(x)\}$. The inverse mapping of $\mathcal{P}$ is denoted by $\mathcal{P}^{-1} : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ and defined as $\mathcal{P}^{-1}(y) = \{x \in \mathbb{R}^n | y \in \mathcal{P}(x)\}$. Note that in the context of set-valued mappings the inverse is always well defined. For the standard inner product in $\mathbb{R}^n$ and the corresponding norm, we write $\langle \cdot \mid \cdot \rangle$ and $|\cdot|$, respectively. A set-valued mapping $\mathcal{P} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is called monotone, if $\langle x^* - y^* \mid x - y \rangle \geq 0$ for all $(x, x^*) \in \text{gr}(\mathcal{P})$ and all $(y, y^*) \in \text{gr}(\mathcal{P})$. We call $\mathcal{P}$ maximal monotone, if $\mathcal{P}$ is monotone and there is no other monotone map $\mathcal{P}' : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ such that $\text{gr}(\mathcal{P}) \subseteq \text{gr}(\mathcal{P}')$ and $\text{gr}(\mathcal{P}) \neq \text{gr}(\mathcal{P}')$. See [2–4] for more details.

## 2. Problem formulation

Given matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, $D \in \mathbb{R}^{m\times m}$ and a set-valued map $\mathcal{M} : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$, we are interested in the (possibly non-smooth) dynamical system

$$\dot{x}(t) = Ax(t) + Bz(t) + u(t) \tag{1a}$$
$$w(t) = Cx(t) + Dz(t) \tag{1b}$$
$$z(t) \in -\mathcal{M}(w(t)). \tag{1c}$$

In this description, $x(t) \in \mathbb{R}^n$ denotes the state variable and $u(t) \in \mathbb{R}^n$ the input at time $t \in \mathbb{R}_{\geq 0}$. We are particularly interested in systems of the form (1) having specific maximal monotonicity properties as will be detailed in the next section. Note that (1) can be perceived as Lur'e-type systems [21] with a set-valued map in the feedback path.

The objective of this paper is to present numerical schemes to construct periodic steady-state solutions (provided they exist) of systems of the form (1) corresponding to periodic input functions $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$. These numerical schemes will be accompanied by formal guarantees that the obtained numerical approximations converge to the exact solution (in an appropriate sense) when the discretization parameters converge to specific values. The latter property is referred to as *consistency* of the numerical scheme.

**Example 1.** In this example, we show that the diode bridge circuit shown in Fig. 1 can be represented by the dynamical system of the form (1). Let $x_1$ be the current through the inductor $L_\text{i}$, $x_2$ be the voltage across the capacitor $C_\text{o}$, and $u$ be a sinusoidal voltage source. Let us assume that $(z_1, w_1)$ and $(z_4, w_4)$ are the (current, voltage) pairs of the diodes in the upper part of the bridge and $(z_2, w_2)$ and $(z_3, w_3)$ are the (voltage, current) pairs of the other diodes. By applying the Kirchhoff