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Optimal Bi-elliptic transfer between two generic coplanar elliptical orbits

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ABSTRACT

In this article, we show the optimal total delta velocity for bi-elliptic and bi-parabolic (through infinity) transfers between non-coaxial boundary orbits. The bi-parabolic transfer is cotangential.

1. Introduction

Many studies have focused on solving the problem of impulsive transfer between given orbits. Hohmann provided critical developments in the theory of impulsive orbital transfers [1]. Another study presented a comprehensive review of works on this problem [2]. Battin presented the geometrical properties of optimal two-impulse transfer [3]. Lawden presented a fundamental result and introduced a primer vector satisfying the necessary conditions for optimality of the total delta velocity [4]. Horner obtained an analytical solution that minimises the total delta velocity impulse between fixed points in the initial and final orbits with a variable transfer angle [5]. Another study [6] showed that in a specific case [5], the optimum two-impulse transfer from an inner orbit to an outer coplanar terminal consisting of a radial distance and a velocity vector is a transfer from the pericentre of the inner orbit. Similarly, if the transfer is from an inner terminal to an outer orbit, the apocentre of the transfer orbit is tangential to the final orbit. One study considered the problem of bi-elliptical transfer between circular coplanar orbits [7]. Another [8] compared Hohmann-type two-impulse trajectories with three-impulse bi-elliptical trajectories to solve the problem of transfer between coaxial orbits with the same directional axes. Herein, we study optimal bi-elliptic transfer with transfer orbits having a fixed apogee. This research is based on the equations presented in Ref. [6]. The obtained result allows us to generalise previous results [7,8] for bi-elliptic transfers between non-coaxial elliptical orbits.

2. Solution of optimal Bi-elliptic transfer problem

The geometrical shape of initial and final orbits is described by their eccentricities and apogee radii. Let $\boldsymbol{r}_1$ and $\boldsymbol{v}_0$ be the position and velocity vector, respectively, at fixed point P with true anomaly f_0. Let $\boldsymbol{r}_2$ and $\boldsymbol{v}_3$ be the position and velocity vector, respectively, at fixed point Q with true anomaly f_3 ($\boldsymbol{v}_0$, $\boldsymbol{v}_3$, $\boldsymbol{r}_1$, and $\boldsymbol{r}_2$ are coplanar). The angle that specifies the separation between the major axes of the initial and final orbits is not fixed. The transfer trajectory consists of two elliptical transfer orbits with the same fixed apogee radius' magnitude r_α and free transfer angles θ_1 and θ_2. The first transfer orbit is formed by applying the impulse $\Delta\boldsymbol{v}_1 = \boldsymbol{v}_1 - \boldsymbol{v}_0$ at point P (velocity becomes $\boldsymbol{v}_1$). The impulse $\Delta\boldsymbol{v}_\alpha = \boldsymbol{v}_{2\alpha} - \boldsymbol{v}_{1\alpha}$ is applied at the apogee of the first transfer orbit to form the second transfer orbit (with same apogee radius $\boldsymbol{r}_\alpha$) which intersects the final orbit at point Q. The impulse $\Delta\boldsymbol{v}_2 = \boldsymbol{v}_3 - \boldsymbol{v}_2$ changes the velocity $\boldsymbol{v}_2$ of the second transfer orbit to velocity $\boldsymbol{v}_3$ at point Q (Fig. 1).

The problem is to choose the transfer orbits such that the total delta velocity.

$$\Delta v_{III\Sigma} = \Delta v_1 + \Delta v_\alpha + \Delta v_2 \quad (1)$$

is minimized. Let impulse Δv_α be applied tangentially at the apogee of the first transfer orbit:

$$\Delta v_\alpha = \sqrt{\mu p_2}/r_\alpha - \sqrt{\mu p_1}/r_\alpha, \quad (2)$$

where μ is the gravitational parameter and p_1 and p_2 are the semi-latera recta of the transfer orbits. In this case, impulse Δv_1 must support the motion ($\cos\varphi_1 > 0$, where φ_1 is the thrust angle at point P (Fig. 2)), and from Ref. [6], we have that its optimal magnitude must be

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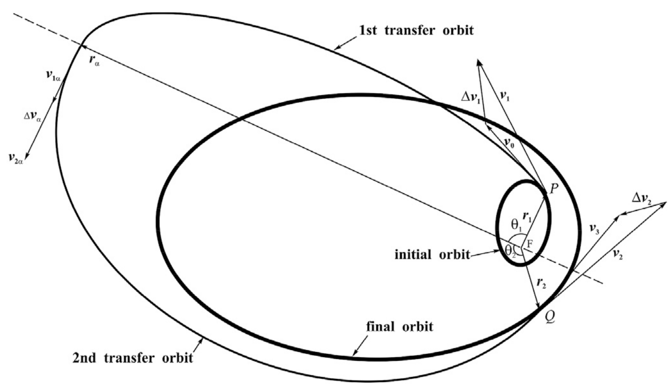

Fig. 1. Scheme of bi-elliptic transfer.

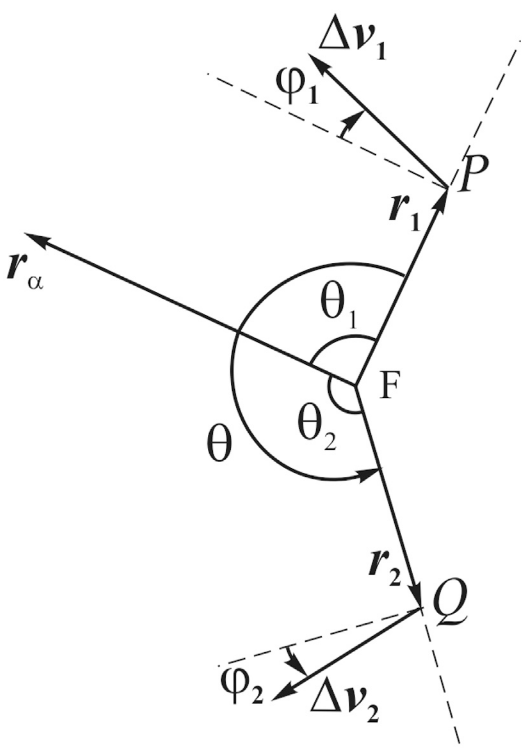

Fig. 2. Scheme of trust angles.

$$\Delta v_1 = \sqrt{\mu p_1}/r_\alpha + \sqrt{2\mu r_\alpha/(r_1+r_\alpha)/r_1} - \sqrt{v_{0r}^2 + \left(v_{0\theta} + \sqrt{2\mu r_1/(r_1+r_\alpha)/r_\alpha}\right)^2}, \tag{3}$$

where v_{0r} is the radial component and $v_{0\theta}$, the transverse in-plane component of vector $\boldsymbol{v}_0$. Therefore, we have a minimum for the delta velocity $\Delta v_1 + \Delta v_\alpha$ of the transfer from the initial orbit to the second transfer orbit. Impulse Δv_2 must oppose the motion ($\cos\varphi_2 < 0$, where φ_2 is the thrust angle at point Q (Fig. 2)), and from Ref. [6], we have that its optimal magnitude must be

$$\Delta v_2 = \sqrt{2\mu r_\alpha/(r_2+r_\alpha)/r_2} - \sqrt{\mu p_2}/r_\alpha - \sqrt{v_{3r}^2 + \left(v_{3\theta} - \sqrt{2\mu r_2/(r_2+r_\alpha)/r_\alpha}\right)^2}, \tag{4}$$

where v_{3r} is the radial component and $v_{3\theta}$, the normal in-plane component of vector $\boldsymbol{v}_3$. By adding Eqs. (2)–(4), we find the solution of the optimal bi-elliptical transfer between coplanar orbits with fixed r_α:

$$\Delta v_{III\Sigma} = \sqrt{2\mu r_\alpha/(r_1+r_\alpha)/r_1} - \sqrt{v_{0r}^2 + \left(v_{0\theta} + \sqrt{2\mu r_1/(r_1+r_\alpha)/r_\alpha}\right)^2} + \sqrt{2\mu r_\alpha/(r_2+r_\alpha)/r_2} - \sqrt{v_{3r}^2 + \left(v_{3\theta} - \sqrt{2\mu r_2/(r_2+r_\alpha)/r_\alpha}\right)^2}. \tag{5}$$

Remarkably, Equation (5) for the total delta velocity does not include the semi-latera recta of the transfer orbits and transfer angles, which are (Eq. (41) and (42) from Ref. [9])

$$p_1 = \frac{2\mu r_1 r_\alpha}{r_1 + r_\alpha}\left(\frac{r_\alpha\cos\varphi_1 - r_1}{r_\alpha - r_1\cos\varphi_1}\right)^2, \quad \tan\frac{\theta_1}{2} = \frac{r_\alpha\cos\varphi_1 - r_1}{(r_\alpha + r_1)\sin\varphi_1}, \tag{6}$$

$$p_2 = \frac{2\mu r_2 r_\alpha}{r_2 + r_\alpha}\left(\frac{r_2 - r_\alpha\cos\varphi_2}{r_\alpha - r_2\cos\varphi_2}\right)^2, \quad \tan\frac{\theta_2}{2} = \frac{r_2 - r_\alpha\cos\varphi_2}{(r_\alpha + r_2)\sin\varphi_2}, \tag{7}$$

where (Eqs. (38) and (39) from Ref. [9]):

$$\tan\varphi_1 = \frac{v_{0r}}{v_{0\theta} + \sqrt{2\mu r_1/(r_1+r_\alpha)/r_\alpha}} \quad (\cos\varphi_1 > 0), \tag{8}$$

$$\tan\varphi_2 = \frac{v_{3r}}{v_{3\theta} - \sqrt{2\mu r_2/(r_2+r_\alpha)/r_\alpha}} \quad (\cos\varphi_2 < 0), \tag{9}$$

3. Analysis

The three-impulse transfer between two elliptical orbits is optimal relative to the $r_{3\alpha}/r_\alpha$ ratio and true anomalies f_0 and f_3. The analysis below describes the conditions under which three-impulse transfer is better than two-impulse transfer.

If $\Delta v_2 = 0$, then the second transfer orbit is the same as the final orbit and $r_\alpha = r_{3\alpha}$ ($r_{3\alpha}$ is the apogee radius of the final orbit). From Eqs. (4) and (5), it follows that

$$\Delta v_{III\Sigma}\big|_{r_\alpha = r_{3\alpha}} = v_{3\alpha} + \sqrt{2\mu r_{3\alpha}/(r_1+r_{3\alpha})/r_1} - \sqrt{v_{0r}^2 + \left(v_{0\theta} + \sqrt{2\mu r_1/(r_1+r_{3\alpha})/r_{3\alpha}}\right)^2}. \tag{10}$$

where $v_{3\alpha}$ is the apogee velocity of the final orbit.

If

$$\left(r_\alpha^2\frac{\partial\Delta v_{III\Sigma}}{\partial r_\alpha}\right)\Bigg|_{r_\alpha=r_{3\alpha}} = \sqrt{\frac{\mu}{2}}\left(\frac{\sqrt{r_1}}{(1+r_1/r_{3\alpha})^{3/2}}\left(1+\cos\varphi_1\left(\frac{r_1}{r_{3\alpha}}+2\right)\right) + \frac{\sqrt{r_2}}{(1+r_1/r_{3\alpha})^{3/2}}\left(1+\cos\varphi_2\left(\frac{r_2}{r_{3\alpha}}+2\right)\right)\right) < 0 \tag{11}$$

(total delta velocity $\Delta v_{III\Sigma}$ decreases when r_α increases and $\cos\varphi_2 < 0$), then we have $r_\alpha > r_{3\alpha}$, that is, bi-elliptic transfer is better than two-impulse transfer (corresponding to $r_2/r_1 > 15.58$ [7] with $\cos\varphi_1 = 1$, $\cos\varphi_2 = -1$, and $r_{3\alpha} = r_2$ for circular boundary orbits).

If $r_\alpha \to \infty$, then bi-elliptic transfer becomes bi-parabolic transfer (also called bi-elliptic through infinity):

$$\Delta v_{\infty\Sigma} = \Delta v_{III\Sigma}\big|_{r_\alpha\to\infty} = \sqrt{\frac{2\mu}{r_1}} - v_0 + \sqrt{\frac{2\mu}{r_2}} - v_3 \tag{12}$$

and

$$p_1\big|_{r_\alpha\to\infty} = 2r_1\cos^2\gamma_0, \quad \tan\frac{\theta_1}{2}\Big|_{r_\alpha\to\infty} = \cot\gamma_1 = \cot\gamma_0, \tag{13}$$