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Neighboring optimal control for fixed-time multi-burn orbital transfers

Zheng Chen

Laboratoire de Mathématiques d'Orsay, Univ. Paris-Sud, CNRS, Université Paris-Saclay, 91405, France

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ABSTRACT

Unlike the classical variational method to design the neighboring optimal control (NOC) by minimizing the second variation of cost functional, this paper, from a geometric point of view, presents a parametric approach to establish the NOC for the problem of fixed-time multi-burn orbital transfers. The key idea of the proposed approach is to construct a parameterized family of neighboring extremals around a reference one. By analyzing the canonical projection of the parameterized family, a geometric interpretation is given to show that, even though the gain matrix for the NOC is not explosive, it is impossible to construct the NOC for the multi-burn problem if conjugate points occur at switching times. Then, through deriving the Taylor series expansion of the parameterized extremals, the neighboring optimal feedback not only on thrust direction but also on switching times is readily devised and the procedure for computing the gain matrix is presented. Moreover, a scheme for performing the NOC is proposed such that the typical singularity when approaching the final time is avoided. Finally, the development of this paper is illustrated by two optimal multi-burn orbital transfer problems.

1. Introduction

The NOC is one of the most important practical applications of optimal control theory as it is fundamental for the neighboring optimal feedback guidance, a well-known NOC-based guidance. As a by-product of deriving the second order conditions (SOC) for optimality, the NOC has been generally derived by minimizing the second variation of cost functional; the references include the pioneering work in [1–5] followed by the more recent work in, e.g., [6–8]. From the variational point of view, the NOC-based guidance for orbital transfers has received quite an extensive treatment (see, e.g., [9–12]). In [9–11], a standing assumption is that the optimal thrust function is continuous. In [12], the NOC for the aeroassisted transfer was investigated. However, to establish the NOC for bang–bang control problems is not a trivial task. Once the optimal thrust is bang–bang, the corresponding NOC consists not only of the feedback on thrust direction but also of that on switching times. Indeed, the NOC for bang–bang control problem has been studied in the literature (see, e.g., [13–15]). In [13] by Mcintyre, to derive the linear feedback on switching times, a rescaling of time variable was used to compensate for the presence of initial errors and disturbing forces. Then, Foerster and Flügge-Lotz [14] extended the result of Mcintyre to the problem of controlling the attitude of satellite. Using a multiple shooting technique, an iterative algorithm for computing the NOC of general optimal control problem with both control and state constraints has been developed in [16], which was then applied to a space shuttle guidance in [17]. To the author's knowledge, the most relevant work about the NOC for the multi-burn problem is the paper [15] by Kornhauser and Lion, who established an accessory minimum problem (AMP) and derived the NOC by solving the AMP. More recently, Chuang et al. [18] studied the NOC of each burn arc without considering the multi-burn trajectory as a whole such that a sub-optimal guidance scheme was developed.

Unlike the various methods developed or used in [13–15], this paper aims to present a compact approach to derive the NOC for multi-burn problem. The key idea of the proposed approach is to parameterize the terminal Lagrangian manifold, defined by the terminal boundary and the corresponding transversality conditions, such that a parameterized family of neighboring extremals around a reference one is constructed. First, by analyzing the canonical projection of the parameterized family of extremals, the disconjugacy conditions (or no-conjugate-point conditions) for the fixed-time multi-burn problem are represented, recalling that conjugate points may occur not only between but also at switching times. According to the *Shadow Price Lemma* in [19,20], the disconjugacy conditions, when met, are sufficient for local optimality provided that each switching point of the reference extremal is regular (cf. Theorem 1).

It is worth mentioning that the existence of neighboring extremals is a prerequisite to establish NOC. For smooth extremals, the existence of neighboring extremals is related to the classical Jacobi necessary condition [3,4]. In fact, the gain matrix for NOC is

E-mail address: zheng.chen@math.u-psud.fr.

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explosive once the Jacobi necessary condition is violated. However, as far as the author knows, the prerequisites for the existence of neighboring extremals around a bang–bang reference one have not been analyzed in the literature. In this paper, a geometric interpretation is presented to show that, even though the gain matrix for the NOC is not explosive, it is still impossible to construct the NOC for the multi-burn problem if conjugate points occur at switching times; that generalizes the result of the classical variational method for smooth extremals (see, e.g., [3,4]).

Assuming the disconjugacy conditions are satisfied along the reference extremal, as shall be shown, the first order term of the Taylor series expansion of costate is a multiplication of a gain matrix and the deviation of the state from the reference trajectory. Since the Pontryagin maximum principle (PMP) [21] gives a natural feedback on control, the NOC for the multi-burn problem is then readily devised in the present paper. Moreover, the numerical procedure for computing the gain matrix is derived, indicating that the gain matrix is discontinuous at switching times. A typical issue in the NOC-based guidance is that the gain matrix is singular when approaching the final time (see, e.g., [4,7,8]). By proposing a scheme for performing the NOC in this paper, such singularity is readily avoided.

The present paper is organized as follows. In Sect. 2, the optimal multi-burn orbital transfer problem is formulated. In Sect. 3, a parameterized family of neighboring extremals around a reference one is constructed. In Sect. 4, the NOC is devised by deriving the first-order term of the Taylor series expansion of the parameterized extremals. In Sect. 5, the numerical procedure for computing the gain matrix is presented. Then, a scheme for performing the NOC is proposed in Sect. 6. In Sect. 7, to illustrate the development of this paper, two finite-thrust fuel-optimal orbital transfers are computed and Monte Carlo tests are performed to test the NOC.

2. Optimal control problem

Consider the spacecraft as a mass point. The state $\boldsymbol{x} \in \mathbb{R}^n$ ($n = 7$) consists of the position vector $\boldsymbol{r} \in \mathbb{R}^3$, the velocity vector $\boldsymbol{v} \in \mathbb{R}^3$, and the mass $m \in \mathbb{R}$, i.e., $\boldsymbol{x} = [\boldsymbol{r}^T, \boldsymbol{v}^T, m]^T$. Then the motion of the spacecraft in an inertial Cartesian coordinate is governed by

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), \rho(t), \boldsymbol{\tau}(t)) := \boldsymbol{f}_0(\boldsymbol{x}(t)) + \rho(t)\boldsymbol{f}_1(\boldsymbol{x}(t), \boldsymbol{\tau}(t)), \quad (1)$$

with

$$\boldsymbol{f}_0(\boldsymbol{x}) = \begin{bmatrix} \boldsymbol{v} \\ -\mu\boldsymbol{r}/\| \boldsymbol{r} \|^3 \\ 0 \end{bmatrix} \text{ and } \boldsymbol{f}_1(\boldsymbol{x}, \boldsymbol{\tau}) = \begin{bmatrix} \boldsymbol{0} \\ u_{\max}\boldsymbol{\tau}/m \\ -\beta u_{\max} \end{bmatrix},$$

where the notation "$\|\cdot\|$" denotes the Euclidean norm, $\mu = 3.986 \times 10^8$ km^3/s^2 is the Earth gravitational constant (the multiplication of the universal gravitational constant and the mass of the Earth), $u_{\max} > 0$ is the maximum magnitude of thrust, $\beta > 0$ is determined by the specific impulse of the engine, $\rho \in [0, 1]$ is the normalized thrust magnitude, and $\boldsymbol{\tau} \in \mathbb{S}^2$ is the unit vector of thrust direction on each burn arc. For the sake of notational clarity, let $\mathcal{X} \subset \mathbb{R}^n$ be the admissible set of $\boldsymbol{x}$. The performance index of the minimum-propellant problem is equivalent to

$$J = \int_0^{t_f} \rho(t)dt, \quad (2)$$

where $t_f > 0$ is the terminal time that is considered fixed in this paper. The initial state is fixed, i.e., $\boldsymbol{x}(0) = \boldsymbol{x}_0$, and the final state $\boldsymbol{x}(t_f)$ takes values in

$$\mathcal{M} := \{\boldsymbol{x} \in \mathcal{X} \mid \phi(\boldsymbol{x}) = 0\}, \quad (3)$$

where $\phi : \mathcal{X} \to \mathbb{R}^l$ ($l \leq n$ is a positive integer) is a twice continuously differentiable function. According to [22], the controllability of the system in Eq. (1) holds in an appropriate subregion of $\mathcal{X}$ for every $u_{\max} > 0$.

Define the costate of $\boldsymbol{x}$ by $\boldsymbol{p} = [\boldsymbol{p}_r^T, \boldsymbol{p}_v^T, p_m] \in T^*_{\boldsymbol{x}}\mathcal{X}$ where $\boldsymbol{p}_r$, $\boldsymbol{p}_v$, and p_m are the costate variables associated with $\boldsymbol{r}$, $\boldsymbol{v}$, and m, respectively. Note that $\boldsymbol{p}$ is a row vector and that $\boldsymbol{p}_r$ and $\boldsymbol{p}_v$ are column vectors. As the abnormal solutions have been ruled out in [23], the Hamiltonian function H can be defined as

$$H = \boldsymbol{p}\boldsymbol{f}_0(\boldsymbol{x}) + \rho\boldsymbol{p}\boldsymbol{f}_1(\boldsymbol{x}, \boldsymbol{\tau}) - \rho =: H_0 + \rho H_1. \quad (4)$$

According to the PMP [21], the optimal thrust direction is along the direction of the primer vector $\boldsymbol{p}_v$ [24], i.e.,

$$\boldsymbol{\tau} = \boldsymbol{p}_v/\|\boldsymbol{p}_v\| \text{ if } \|\boldsymbol{p}_v\| \neq 0. \quad (5)$$

Substituting Eq. (5) into Eq. (4), we have that the switching function

$$H_1 := \frac{\|\boldsymbol{p}_v\|}{m}u_{max} - p_m\beta u_{max} - 1$$

determines the switching between burn and coast arcs:

$$\rho = \begin{cases} 1 & H_1 > 0, \\ 0 & H_1 < 0. \end{cases} \quad (6)$$

Singular thrust arcs ($0 < \rho < 1$) are not considered here since they have been shown to be non-optimal in general [25]. Note that the optimal control variables are functions of state and costate variables. The state and costate are governed by

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \frac{\partial H}{\partial \boldsymbol{p}}(\boldsymbol{x}(t), \boldsymbol{p}(t)), \\ \dot{\boldsymbol{p}}(t) = -\frac{\partial H}{\partial \boldsymbol{x}}(\boldsymbol{x}(t), \boldsymbol{p}(t)), \end{cases} \quad (7)$$

and the transversality condition is

$$\boldsymbol{p}(t_f) = \boldsymbol{\nu}\nabla\phi(\boldsymbol{x}(t_f)), \quad (8)$$

where the time-independent row vector $\boldsymbol{\nu} \in (R^l)^*$ denotes Lagrangian multipliers. Note that the transversality condition indicates $\boldsymbol{p}(t_f) \perp T_{\boldsymbol{x}(t_f)}\mathcal{M}$.

3. Parameterization of neighboring extremals

Hereafter, we denote by $(\boldsymbol{x}(\cdot), \boldsymbol{p}(\cdot)) : [0, t_f] \to T^*\mathcal{X}$ the solution of Eq. (7) with the maximum conditions in Eqs. (5), (6) satisfied. Such a solution $(\boldsymbol{x}(\cdot), \boldsymbol{p}(\cdot))$ on $[0, t_f]$ is said an extremal. Furthermore, we denote by $(\bar{\boldsymbol{x}}(\cdot), \bar{\boldsymbol{p}}(\cdot))$ on $[0, t_f]$ the reference (or nominal) extremal which satisfies the boundary and transversality conditions: $\bar{\boldsymbol{x}}(0) = \boldsymbol{x}_0$, $\bar{\boldsymbol{x}}(t_f) \in \mathcal{M}$, and $\bar{\boldsymbol{p}}(t_f) \perp T_{\bar{\boldsymbol{x}}(t_f)}\mathcal{M}$.

Definition 1 *(Neighboring extremals).* Let $\mathcal{W} \subset T^*\mathcal{X}$ be a small tubular neighborhood of the reference extremal $(\bar{\boldsymbol{x}}(\cdot), \bar{\boldsymbol{p}}(\cdot))$ on $[0, t_f]$. Then, we say every extremal $(\boldsymbol{x}(\cdot), \boldsymbol{p}(\cdot))$ on $[0, t_f]$, with the final conditions: $\boldsymbol{x}(t_f) \in \mathcal{M}$ and $\boldsymbol{p}(t_f) \perp T_{\boldsymbol{x}(t_f)}\mathcal{M}$, is a neighboring extremal of the reference one if $(\boldsymbol{x}(\cdot), \boldsymbol{p}(\cdot))$ on $[0, t_f]$ lies in $\mathcal{W}$.

Note that the initial point of every neighboring extremal is not restricted and that the final point satisfies the boundary condition in Eq. (3) and the corresponding transversality condition in Eq. (8). In the next paragraph, the neighboring extremals will be parameterized.

First of all, we make a regular assumption on the final constraint function.

Assumption 1. The matrix $\nabla\phi(\bar{\boldsymbol{x}}(t_f))$ is of full rank.