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Analytical benchmark for linear wave scattering by a submerged circular shoal in the water from shallow to deep

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ABSTRACT

An analytical study on linear wave propagation over a submerged circular shoal is conducted, where the water depth is quasi-idealized with the water depth within the shoal region being a positive constant plus a power function of the radial distance. By using two variable transforms, a new analytical solution in the form of Frobenius series to the modified mild-slope equation (MMSE) is constructed and the convergence condition of the series is analyzed and clarified. The present solution extends the validity scope of the existing analytical solutions for wave propagation over a circular shoal from the long-wave range to the whole wave range. Comparison among the present solution, two sets of experimental solutions and MMSE based numerical solutions is conducted and nice agreements is obtained. Based on the present model, the influence of shoal size such as the radius, height and concavity on wave amplification is investigated. It is shown that the maximal wave amplification increases with the increase of the shoal size. The study of the influence of the incident wavelength on wave amplification shows that the maximal wave amplification often occurs when the wavelength is about five times of the global water depth.

1. Introduction

When propagating over a submerged shoal such as a coral reef or artificial fish reef in coastal region, water waves may be transformed and amplified because of the scattering effect including reflection, refraction and diffraction caused by the variation of water depth. Due to the physical and practical significance, the related problem has been intensively studied for many decades by using numerical models (Berkhoff, 1972; Flokstra and Berkhoff, 1977; Radder, 1979; Panchang et al., 1988, 1990; Zhu, 1993; Chandrasekera and Cheung, 1997; Choi et al., 2009; Choi and Yoon, 2011; Griffiths and Porter, 2012) or by experimental models (Sharp, 1968; Vincent and Briggs, 1989; Lie and Tϕrum, 1991; Suh et al., 2001; Lee et al., 2013). Analytical models are very welcome due to their high accuracy, low cost in labor, time saving and important role in theoretical analysis and benchmark test to numerical models. However, due to the difficulty in solving differential equations, analytical solutions are generally hard to obtain.

The first analytical solution for wave propagation over a shoal was given by Zhang and Zhu (1994), who considered a circular paraboloidal shoal where the water depth within the shoal region is assumed to be $h(r) = h_1 + (h_0 - h_1)(r/r_0)^2$, with r being the radial distance from the center of the shoal, h_0 the global water depth over the flat bottom outside the shoal, and r_0 and h_1, respectively, the radius and submergence of the shoal. They solved the linear long-wave equation (LWE) and constructed an analytical solution in terms of Frobenius series which unconditionally converges in the whole shoal region.

Liu and Li (2007) considered wave propagation over a circular truncated shoal with the variable water depth over the shoal being assumed to be $h(r) = h_0(r/r_0)^s$, which is a power function of the radial distance. Bottom topographies with such kind of depth profile were classified as idealized seabeds by Liu et al. (2013). Due to this idealized assumption, the truncated shoal crest is exactly located on the still water level and an analytical solution to the LWE in a closed-form in terms of Bessel functions can be obtained. For the same reason, most of the variable seabed topographies considered in previous analytical studies to the LWE for linear wave propagation were also assumed to be idealized, see Kajiura (1961), Dean (1964) and Liu et al. (2013) in two dimensional case and Homma (1950), Yu and Zhang (2003) and Niu and Yu (2011a, 2011b) in three dimensional case.

Zhu and Harun (2009) solved the LWE again for wave propagation over the same paraboloidal shoal studied by Zhang and Zhu (1994), in which Suh et al.'s (2005) analytical technique in solving long wave

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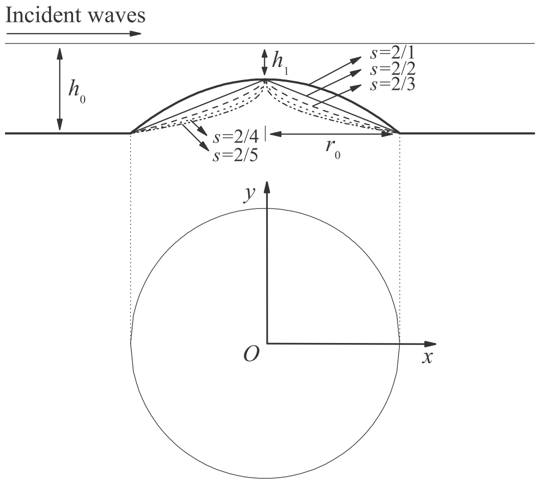

Fig. 1. A definition sketch of a submerged circular shoal.

scattering by a circular bowl pit was employed. However, due to the difference between the bowl pit and the paraboloidal shoal, the Frobenius series constructed by Suh et al. (2005) unconditionally converges in the whole pit region, while the one constructed by Zhu and Harun (2009) converges only when the shoal is sufficiently submerged with $h_1/h_0 > 0.5$, in fact, in Zhu and Harun (2009, lines 4–5 on p. 318), the condition $a > b$ is equivalent to $h_1/h_0 > 0.5$.

Niu and Yu (2011c) further extended Zhu and Harun's (2009) solution for a paraboloidal shoal to a much more general shoal with the water depth over the shoal being modified to $h(r) = h_1 + \beta r^s$, where $\beta = (h_0 - h_1)/r_0^s$ and s is an arbitrary positive real number. They constructed an analytical solution in the form of Frobenius series to the LWE. However, their solution is also convergent only if the condition $h_1/h_0 > 0.5$ is satisfied, see Eq. (27) in Niu and Yu (2011c).

To remove the restriction on the shoal submergence in both Zhu and Harun (2009) and Niu and Yu's (2011c) series solutions, Liu and Xie (2011) introduced a new variable transform $t = \beta r^s/(\beta r^s + h_1)$, see their Eq. (6). Based on the key transform, a Frobenius series with unconditional convergence in the entire shoal region with any submergence was constructed. Liu and Xie (2011) further showed that, their new solution in the limiting case with the power exponent s approaching the infinity degenerates into the classical analytical solution for wave propagation over a submerged circular cylinder given by Longuet-Higgins (1967).

However, as the names "long-wave equation" and "shallow-water equation" indicate, the validity scope of all the aforementioned analytical solutions is restricted to the long waves or the shallow-water waves. The first analytical solution for wave scattering by a submerged truncated paraboloidal shoal with the validity scope being the whole wave spectrum from shallow water to deep water was constructed by Lin and Liu (2007), where the traditional mild-slope equation (MSE, Berkhoff, 1972) was solved by employing an approximate analytical technique proposed by Liu et al. (2004). To the authors' knowledge, exact or approximate analytical solution for a un-truncated shoal with the validity scope being the whole wave spectrum has never been constructed though several experiments for wave scattering by circular paraboloidal shoals were conducted by Sharp (1968) (also see Williams et al., 1980) and Suh et al. (2001).

In this paper, we are going to construct an exact analytical solution to the modified mild-slope equation (MMSE, Chamberlain and Porter, 1995) for linear wave scattering by a general un-truncated shoal where $h(r) = h_1 + \beta r^s$ with $2/s$ being a positive integer S, i.e., $s = 2/S$, $S = 1, 2, 3, \ldots$ Hence those paraboloidal shoals (i.e., $s = 2$) experimentally studied by Sharp (1968) and by Suh et al. (2001) are included in the present study. Because the present solution is based on the MMSE, it is not only valid in the whole spectrum from shallow water to deep water, but can also deal with those shoals with relatively steep surfaces.

2. Series solution and convergence analysis

As shown in Fig. 1, we consider the scattering of simple harmonic waves over a submerged circular shoal, where r_0, h_0 and h_1 are the radius of the shoal, the constant water depth over the flat bottom, and the submergence of the shoal, respectively. Let x, y and z be Cartesian coordinates with its origin being located above the shoal center on the still water level, then the water depth in the whole region is given by

$$h(r) = \begin{cases} h_0, & r \in (r_0, +\infty), \\ h_1 + \beta r^s, & r \in [0, r_0], \end{cases} \tag{1}$$

with $r = \sqrt{x^2 + y^2}$, $\beta = (h_0 - h_1)/r_0^s$, and $2/s$ being a positive integer S, i.e., $s = 2/S$, $S = 1, 2, 3, \ldots$

According to Chamberlain and Porter (1995), the water surface elevation $\eta = \eta(x, y)$ may satisfy the following MMSE (a later version of the MMSE see Porter, 2003)

$$\nabla \cdot (u_0 \nabla \eta) + \left[k^2 u_0 + u_1 \nabla^2 h + u_2 (\nabla h)^2\right] \eta = 0, \tag{2}$$

where

$$u_0(r) = \frac{1}{2k} \tanh kh \left(1 + \frac{2kh}{\sinh 2kh}\right), \tag{3}$$

$$u_1(r) = \frac{\text{sech}^2 kh}{4(2kh + \sinh 2kh)} (\sinh 2kh - 2kh \cosh 2kh), \tag{4}$$

$$u_2(r) = \frac{k\,\text{sech}^2 kh}{12(2kh + \sinh 2kh)^3} \left[16(kh)^4 + 32(kh)^3 \sinh 2kh - 9 \sinh 2kh \sinh 4kh + 6kh(2kh + 2\sinh 2kh)\left(\cosh^2 2kh - 2\cosh 2kh + 3\right)\right], \tag{5}$$

∇ is the horizontal gradient operator, and k is the wave number determined by the following implicit dispersion relation

$$\omega^2 = gk \tanh kh, \tag{6}$$

in which ω is the angular frequency and g is the gravitational acceleration. The dispersion relationship was first derived by Airy in 1841, see Craik (2004), and is valid for any water depth from shallow water to deep water, laying a solid foundation for modern hydrodynamics and ocean engineering (Lamb, 1932; Dean and Dalrymple, 1984; Mei, 1989).

Since the seabed is axisymmetrical, we better introduce the cylindrical coordinates (r, θ) with $x = r\cos\theta$ and $y = r\sin\theta$. It is noted that u_0 is independent of θ, then Eq. (2) can be rewritten as

$$\frac{\partial^2 \eta}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 \eta}{\partial^2 \theta} + \left(\frac{\mathrm{d}\ln u_0}{\mathrm{d}r} + \frac{1}{r}\right)\frac{\partial \eta}{\partial r} + \left[k^2 + \frac{u_1}{u_0}\left(\frac{\mathrm{d}^2 h}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}h}{\mathrm{d}r}\right) + \frac{u_2}{u_0}\left(\frac{\mathrm{d}h}{\mathrm{d}r}\right)^2\right]\eta = 0. \tag{7}$$

Further, using the variable separation technique, we may expand a solution to Eq. (7) into a Fourier-cosine series as

$$\eta(r, \theta) = \sum_{n=0}^{\infty} R_n(r) \cos n\theta. \tag{8}$$

By substituting Eq. (8) into Eq. (7) and using the orthogonal property of function series $\{\cos n\theta\}$, Eq. (7) is reduced into the following ordinary differential equation

$$R''_n(r) + A(r) R'_n(r) + B(r) R_n(r) = 0, \tag{9}$$