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Fast solution of neutron diffusion problem by reduced basis finite element method

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A R T I C L E I N F O

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For the important applications which need carry out many times of neutron diffusion calculations such as the fuel depletion analysis and the neutronics-thermohydraulics coupling analysis, fast and accurate solutions of the neutron diffusion equation are demanding but necessary. In the present work, the certified reduced basis finite element method is proposed and implemented to solve the generalized eigenvalue problems of neutron diffusion with variable cross sections. The order reduced model is built upon high-fidelity finite element approximations during the offline stage. During the online stage, both the k_{eff} and the spatial distribution of neutron flux can be obtained very efficiently for any given set of cross sections. Numerical tests show that a speedup of around 1100 is achieved for the IAEA two-dimensional PWR benchmark problem and a speedup of around 3400 is achieved for the three-dimensional counterpart with the fission cross-sections, the absorption cross-sections and the scattering cross-sections treated as parameters.

1. Introduction

The multi-group diffusion approximation for the transport of neutrons has been widely used in reactor analysis. Through introducing some hypothesis such as that the neutrons can be grouped into energy ranges (groups) and that the distribution of the neutron velocity vectors is independent of the direction, the neutron transport equation can be reduced to a group of much simpler diffusion equations and several techniques such as the modern nodal methods(Cho, 2005), the finite difference method and the finite element method (Kang and Hansen, 1973; Kavenoky and Lautard, 1982; Adams and Martin, 1992; Wareing et al., 2001), have been well developed to solve the partial differential equations. Considering a coarse mesh ($\sim$10 cm) is usually sufficient for such techniques, the computational cost is generally low and acceptable for routine analysis. However, high requirement for computation efficiency is yet to be satisfied for some important applications which need carry out many times of calculations such as fuel depletion analysis, three dimensional core analysis with movement of control rods or the neutronics-thermohydraulics coupling analysis. Challenges are even demanding if more elaborate approximations, such as the simplified P_N (i.e., SP_N) method, are adopted. It is interesting to note that although the neutron diffusion equations have to be solved many times in these applications, the mathematical form of the equations remains the same and only the coefficients (i.e., the macro cross-sections) vary. If the varying coefficients are treated as parameters, the neutron diffusion equations can thus be parametrized and the computational cost can be effectively reduced by some model reduction techniques such as the proper orthogonal decomposition, the adaptive cross approximation, the empirical interpolation method and the reduced basis finite element (Quateroni and Rozza, 2014).

Among the various reduced order methods (ROMs), the certified reduced basis finite element method (RB-FEM) (Hesthaven et al., 2016; Quarteroni et al., 2016) has witnessed a spectacular effervescence in the past decade. Its high efficiency as well as the guaranteed accuracy are ensured by a full decoupling of the finite element scheme and the reduced order model through an offline-online procedure. The complexity of the offline stage depends on the complexity of the finite element approximation of the parametrized partial differential equation, while the complexity of the online stage depends solely on the complexity of the reduced order model. When combined with the posteriori error estimation, the online stage guarantees the accuracy of the reduced order model. Sartori et al. (2016) has successfully applied the RB-FEM to simulate the nuclear reactor control rods movement and obtained a fast-running prediction of reactivity and neutron flux distribution with the geometric domain parametrized. However, the application of

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the promising method is still rare in the domain of nuclear engineering.

In the present work, we apply the RB-FEM to solve the generalized eigenvalue problem with the neutron diffusion equations parametrized by the macro cross-sections. A reduced set of basis is chosen by using the greedy sampling strategy. And then approximate but sufficiently accurate solutions can be evaluated quickly for any given set of cross sections. Two standard examples, i.e., the IAEA two-dimensional and three-dimensional pressurized water reactor (PWR), are used to test and examine the built order reduced model.

2. Parametrized neutron diffusion equation

Only the stationary multi-group neutron diffusion equation is considered in the study,

$$-\nabla\cdot(D_g\cdot\nabla\Phi_g)+\sum\nolimits_{r,g}\cdot\Phi_g=\sum_{g'\neq g}^{G}\sum_{s,g'\to g}\cdot\Phi_{g'}+\frac{\chi_g}{K_{eff}}\cdot\sum_{g'=1}^{G}\upsilon\sum_{f,g'}\cdot\Phi_{g'}+S_{ext,g}\qquad(1)$$

For group g: Φ_g is the neutron flux, D_g is the diffusion coefficient, $\sum_{f,g}$ is the fission cross section, υ is the number of neutrons emitted per fission, χ_g is the fission spectrum, $\sum_{r,g}$ is the removal cross section, and $\sum_{s,g'\to g}$ is the scattering cross section from group g to group $g\prime$ and K_{eff} is the effective multiplication factor. In the present study, only the fission cross section, the removal cross section and the scattering cross section are treated as parameters considering they are most sensitive to the burnup, the temperature of the fuel and the temperature of the coolant. Thus the parameter vector can be represented by $\mu=\left[\sum_{r,g},\sum_{f,g},\sum_{s,g'\to g}\right]$.

The equation above is augmented by the albedo boundary condition,

$$D_g\nabla\Phi_g\cdot\boldsymbol{n}(r)+\frac{1}{2}\frac{1-\beta}{1+\beta}\Phi_g=0\qquad(2)$$

where $\boldsymbol{n}(r)$ denotes the normal direction of the boundary(Fig. 1). Reflectivity β is defined as the ratio of the incoming current with the outgoing current:

$$\beta=\frac{J_g^-}{J_g^+}\qquad(3)$$

To represent the vacuum boundary condition, the reflectivity is usually set to be a small non-zero value, i.e. β = 0.031758 (Hébert, 2010). To represent a zero incoming current condition, β should be set to be 1.

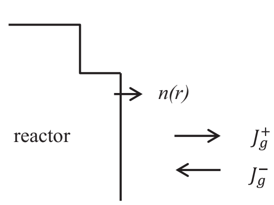

Fig. 1. Schematic of the boundary condition.

3. RB method for generalized eigenvalue problems

3.1. Finite element Formulation of the generalized eigenvalue problem

Ignoring the external neutron source and the weak form of Eq. (1) can be written as,

$$\int_\Omega D_g\cdot\nabla\Phi_g\cdot\nabla v d\Omega+\int_\Omega\sum_{rg}\Phi_g v d\Omega-\int_{\partial\Omega}D_g\cdot\frac{\partial\Phi_g}{\partial\boldsymbol{n}}\cdot v d\partial\Omega$$
$$=\sum_{g'\neq g}^{G}\int_\Omega\sum_{sg'\to g}\Phi_g v d\Omega+\frac{\chi_g}{K_{eff}}\sum_{g'=1}^{G}\int_\Omega\upsilon\sum_{f,g'}\cdot v d\Omega\qquad(4)$$

where Ω and $\partial\Omega$ represent the domain of the problem and the corresponding boundary, respectively.$v\in V$ is the test function and V is a Hilbert space with an induced norm $\|\cdot\|_V=\sqrt{(\cdot,\cdot)_V}$.

The approximate solution of the neutron flux can be expressed as the combination of the basis functions $\{\varphi_i, i=1,2,\ldots,N_v\}$, i.e., $\Phi_g=\sum_{j=1}^{N_v}u_j^g\cdot\varphi_j$ where u_j^g is the coefficient to be sought and N_v is the number of basis functions. According to the Galerkin finite element method, the test function can be expressed as $v=\sum_{j=1}^{N_v}b_j\cdot\varphi_j$ where b_j is an arbitrary constant. Substitute the discrete forms of Φ_g and v into Eq. (4) and notice the arbitrariness of b_j, one has,

$$\chi_g\sum_{g'=1}^{G}\sum\nolimits_{j=1}^{N_v}u_j^{g'}\int_\Omega(\upsilon\sum_{f,g'}\varphi_j\cdot\varphi_i)d\Omega$$
$$=K_{eff}\sum\nolimits_{j=1}^{N_v}u_j^g\left(\int_\Omega\left(D_g\cdot\nabla\varphi_j\cdot\nabla\varphi_i+\sum\nolimits_{r,g}\varphi_j\cdot\varphi_i\right.\right.$$
$$\left.\left.-\sum\nolimits_{g'\neq g}^{G}\sum\nolimits_{s,g'\to g}\varphi_j\cdot\varphi_i-\right)d\Omega-\int_{\partial\Omega}D_g\frac{\partial\varphi_j}{\partial n}\varphi_j\cdot\varphi_i d\partial\Omega\right)\qquad(5)$$

Let $\sum_{r,g}=\sum_{a,g}+\sum_{g'\neq g}^{G}\sum_{s,g\to g'}$and incorporate the albedo boundary condition (i.e., Eq. (2)),

$$\chi_g\sum\nolimits_{g'=1}^{G}\sum\nolimits_{j=1}^{N_v}u_j^{g'}\int_\Omega\left(\upsilon\sum\nolimits_{f,g'}\varphi_j\cdot\varphi_i\right)d\Omega$$
$$=K_{eff}\sum\nolimits_{j=1}^{N_v}u_j^g\left(\int_{\partial\Omega}\frac{1}{2}\frac{1-\beta}{1+\beta}\varphi_j\cdot\varphi_i d\partial\Omega+\int_\Omega\left(D_g\cdot\nabla\varphi_j\cdot\nabla\varphi_i\right.\right.$$
$$\left.\left.+\sum\nolimits_{a,g}\varphi_j\cdot\varphi_i+\sum\nolimits_{g'\neq g}^{G}\sum\nolimits_{s,g\to g'}\varphi_j\cdot\varphi_i-\sum\nolimits_{g'\neq g}^{G}\sum\nolimits_{s,g'\to g}\varphi_j\cdot\varphi_i\right)d\Omega\right)\qquad(6)$$

In the case of a two-group problem, i.e., $G=2$, Eq. (6) can be written in the matrix form as,

$$Au=K_{eff}Bu\qquad(7)$$

where A and B can be affinely decomposed as,

$$A=\begin{pmatrix}A_{11}&A_{12}\\A_{21}&A_{22}\end{pmatrix}=\upsilon\sum_{f,1}\begin{pmatrix}A_{11}&0\\A_{21}&0\end{pmatrix}+\upsilon\sum_{f,2}\begin{pmatrix}0&A_{12}\\0&A_{22}\end{pmatrix}\qquad(8)$$

$$B=\begin{pmatrix}B_{11}+D&0\\0&B_{22}+D\end{pmatrix}+\sum_{s,2\to1}\begin{pmatrix}0&B_{12}\\0&0\end{pmatrix}+\sum_{s,1\to2}\begin{pmatrix}0&0\\B_{21}&0\end{pmatrix}\qquad(9)$$

$$u=\left(u_1^1,u_2^1,\ldots,u_{N_V}^1,u_1^2,u_2^2,\ldots,u_{N_V}^2\right)^T\in R^{2N_V\times1}\qquad(10)$$

$$A_{11}(\mathrm{i,j})=A_{12}(\mathrm{i,j})=\int_\Omega\chi_1(\varphi_j\cdot\varphi_i)d\Omega\qquad(11)$$

$$A_{21}(\mathrm{i,j})=A_{22}(\mathrm{i,j})=\int_\Omega\chi_2(\varphi_j\cdot\varphi_i)d\Omega\qquad(12)$$

$$B_{11}(\mathrm{i,j})=\int_\Omega(D_1\cdot\nabla\varphi_j\cdot\nabla\varphi_i+\sum\nolimits_{r,1}\varphi_j\cdot\varphi_i)d\Omega\qquad(13)$$