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Diffusion theory model of the neutron number probability distribution in a subcritical multiplying assembly

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ABSTRACT

The probability distribution of neutron numbers in a symmetric subcritical reflected fissile sphere is numerically obtained using a one-speed diffusion approximation to the underlying backward Master equation. Employing an accurate space–time discretisation scheme, the coupled but closed system of equations for the number probabilities is sequentially solved as a function of position and time of an injected neutron. This solution is then used to construct the corresponding distributions for a random intrinsic source of arbitrary multiplicity. Numerical results clearly demonstrate the importance of including spatial dependence in the neutron number probability distributions, which show complex spatial behaviours that cannot be encapsulated in a point model system. This is especially evident in the case of the multi-region model where the presence of a reflector is seen to alter the approach to steady state of the number probabilities, while the material interface has a significant effect on the magnitude of the probabilities, both locally and globally.

1. Introduction

Measurement and modelling of neutron multiplicity distributions play a key role in the identification and characterisation of small samples of special nuclear material that is important in the nonproliferation and safeguards space (Böhnel, 1985; Enqvist et al., 2006; Mattingly, 2012). A knowledge of the neutron multiplicity distribution makes possible the estimation of integral properties such the strength of the inherent random source, and the mass and composition of an unknown sample (Mattingly, 2012; Enqvist et al., 2006). Moreover, because the neutronic signature of fissile materials deviates from the Poisson distribution, multiplicity distributions can help distinguish background neutrons arising from (α, n) reactions from those arising from fission chains.

Computational analysis of neutron counting in nuclear safeguards, with detailed physical geometry representation and accounting for energy dependence of neutron interactions and source spectra, has largely relied on either analog Monte Carlo simulation of the branching process (Pozzi et al., 2003), which is expensive and primarily useful for benchmarking purposes, or on the construction of low-order statistical moments using deterministic codes to obtain parameters such as the Feynmann-Y counting statistic from which integral properties of the sample can be inferred (Mattingly, 2012). An alternative approach is provided by the underlying Master equation for the probability distribution of neutron number under general conditions, but the complexity of this formulation makes numerical solution intractable for distributions of large neutron numbers. However, as the Master equation is equivalent to a coupled but closed set of equations for the neutron number distribution of successively higher order, direct numerical solution becomes feasible if the order of the neutron number necessary to adequately characterise the distribution is not high, typically in the tens. These are precisely the conditions that arise when dealing with small, highly subcritical samples and hence the approach is relevant to nonproliferation and safeguards work. Thus far, however, numerical work based on the Master equation has been restricted to computing neutron and photon distributions in lumped or point systems for a single source realisation (Enqvist et al., 2006). While useful for developing insight into the statistics of neutron multiplication and quantifying the effect of random sources, lumped models are unable to describe spatial effects originating from nonuniformity in the distribution of fissile material and the presence of neutron reflectors and absorbers. To describe such nonlocal phenomena it is necessary to treat spatial dependence explicitly in the Master equation. The objective of this work is to develop such a generalisation using the one-speed diffusion

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approximation and demonstrate the feasibility of obtaining numerical solutions for neutron numbers of varying orders in planar and spherical geometries. We conclude this discussion on numerical methods by mentioning other methods which have been developed to compute the neutron number probability distributions. Work by Prasad and Snyderman (2012) developed a recursive point model for subcritical systems regarding the time-asymptotic neutron number probability distribution of neutrons which escape a multiplying medium and are therefore available for detection. Additionally, Abate and Whitt (1992), Williams (2016) and Chambers et al. (2016) have used numerical inversion of the generating function transform to obtain the neutron number probability distributions. However, this technique has only been applied to lumped models and requires the generating function to be known in closed form. It is not yet clear if this approach can be efficiently extended to handle non-lumped systems.

In this paper, we develop a time dependent, spatially extensive backward model for the neutron number probabilities in the one-speed diffusion approximation and investigate it in both the case of a single initiating neutron and additionally a constantly emitting source. The distributions obtained are benchmarked using semi analytical solutions for special cases in the slab model for the single initiating neutron case. The numerical scheme is then applied to a one dimensional symmetric reflected fissile sphere containing an intrinsic random source of general multiplicity. Extensive numerical results are presented and discussed for the neutron number distributions in various subregions of the system, for different neutron injection locations as well as with a distributed source, that demonstrate unequivocally the importance of explicitly describing spatial heterogeneities in complex multiplying media with weak sources.

2. Backward master equation formulation

We consider a convex multiplying body of volume $V \subset \mathbb{R}^3$ with bounding surface $\partial D \subset \mathbb{R}^2$ in which the neutron population evolves stochastically with time subject to random scattering, parasitic capture, and fission events with known probability laws. Restricting considerations to single energy neutrons, so that the neutron phase space is characterised by spatial position $\vec{r} \in V$ and direction of travel $\vec{\Omega} \in S^2$, we define $p_n(R, t_f; \vec{r}, \vec{\Omega}, t), n = 0, 1, 2 \ldots$, as the probability of finding n neutrons in phase space region $R \subset V \times S^2$ at a time t_f conditioned on a single neutron existing in the body at an earlier time t at position $\vec{r}$ with direction vector $\vec{\Omega}$. Then, under the reasonable assumption that the neutron population evolves as a Markov process, $p_n(\cdots)$ satisfies the Chapmann–Kolmogorov equation or, when expressed in differential form, the more commonly known backward Master equation (Pál, 1962; Bell, 1965; Pázsit and Pál, 2008). The backward Master equation is an open system of coupled nonlinear integro-differential equations for the number distribution, and is more useful when expressed in terms of the probability generating function $G(R, t_f, z; \vec{r}, \vec{\Omega}, t)$ defined by the z-transform (Bell, 1965):

$$G(R, t_f, z; \vec{r}, \vec{\Omega}, t) = \sum_{n=0}^{\infty} z^n p_n(R, t_f; \vec{r}, \vec{\Omega}, t), \quad (1)$$

or, more conveniently, in terms of the modified generating function:

$$\widetilde{G}(R, t_f, z; \vec{r}, \vec{\Omega}, t) = 1 - G(R, t_f, z; \vec{r}, \vec{\Omega}, t). \quad (2)$$

Suppressing the explicit dependence on R and t_f for notational convenience, the governing equation for the generating function is given by Bell (1965):

$$-\frac{1}{v}\frac{\partial \widetilde{G}(z; \vec{r}, \vec{\Omega}, t)}{\partial t} - \vec{\Omega}.\nabla \widetilde{G}(z; \vec{r}, \vec{\Omega}, t) + \Sigma_t(\vec{r}, t)\widetilde{G}(z; \vec{r}, \vec{\Omega}, t)$$
$$= \bar{\nu}(\vec{r}, t)\Sigma_f(\vec{r}, t)\left[\int \widetilde{G}(z; \vec{r}, \vec{\Omega}', t)\frac{d\vec{\Omega}'}{4\pi}\right] + \Sigma_s(\vec{r}, t)\int \widetilde{G}(z; \vec{r}, \vec{\Omega}', t)\frac{d\vec{\Omega}'}{4\pi}$$
$$-\sum_{k=2}^{K}\frac{(-1)^k}{k!}\chi_k(\vec{r}, t)\Sigma_f(\vec{r}, t)\left[\int \widetilde{G}(z; \vec{r}, \vec{\Omega}', t)\frac{d\vec{\Omega}'}{4\pi}\right]^k, \ t \leqslant t_f, \quad (3)$$

where Σ_t is the total cross section which is the sum of scattering, Σ_s, capture, Σ_c, and fission, Σ_f, cross sections. With $p_k, k = 0, 1, \ldots K$, defined as the probability that k neutrons emerge from a fission event, in other words, the neutron multiplicity, the factorial moments are given by $\chi_k = \sum_{i=0}^{K} i(i-1)(i-2)\ldots(i-k+1)p_i$ and are defined such that $\chi_2/2!$ is the mean number of pairs, $\chi_3/3!$ is the mean number of triplets and so on. The nonlinear terms in Eq. (3), of highest degree equal to the maximum number of neutrons K emitted in fission, reflect the underlying branching process and is a source of significant complication for numerical solutions of the backward model, which will be discussed at length later in this paper. In addition to single energy neutrons and the assumption of isotropic scattering, it is assumed that fission neutrons are uncorrelated in angle.

The auxiliary conditions on the neutron number distribution for a single initiating neutron comprise the final time condition:

$$p_n(R, t_f; \vec{r}, \vec{\Omega}, t_f) = \delta_{n,1}\mathcal{I}_R(\vec{r}, \vec{\Omega}) + \delta_{n,0}[1 - \mathcal{I}_R(\vec{r}, \vec{\Omega})], \quad (4)$$

where $\mathcal{I}_R(\vec{r}, \vec{\Omega})$ is the indicator function defined to be unity for $(\vec{r}, \vec{\Omega}) \in R$ and zero otherwise, and the boundary condition:

$$p_n(R, t_f; \vec{r}, \vec{\Omega}, t) = \delta_{n,0}, \quad \vec{r} \in \partial D, \ \vec{e} \cdot \vec{\Omega} > 0, \quad (5)$$

where $\vec{e}$ is the outward unit normal on the system boundary. Using the definition of the generating function, the corresponding final time and boundary conditions on $\widetilde{G}$ are readily determined to be:

$$\widetilde{G}(z; \vec{r}, \vec{\Omega}, t_f) = (1 - z)\mathcal{I}_R(\vec{r}, \vec{\Omega}), \quad (6)$$

and:

$$\widetilde{G}(z; \vec{r}, \vec{\Omega}, t) = 0, \quad \vec{r} \in \partial D, \vec{e} \cdot \vec{\Omega} > 0, \ t < t_f. \quad (7)$$

We note from Eq. (1) that $G(R, t_f, z = 0; \vec{r}, \vec{\Omega}, t) = p_0(R, t_f; \vec{r}, \vec{\Omega}, t)$ represents the extinction probability, that is, the probability of finding no neutrons in R at a time t_f, given that a single neutron was introduced to the system at an earlier time t, with position $\vec{r}$ and direction vector $\vec{\Omega}$. It follows that $\widetilde{G}(R, t_f, z = 0; \vec{r}, \vec{\Omega}, t) = 1 - p_0(R, t_f; \vec{r}, \vec{\Omega}, t)$ is the survival probability, that is, the probability of finding a non-zero number of neutrons in R at a time t_f given one initial neutron. Taken in the limit $t \to \infty$, the survival probability gives rise to the probability of initiation (POI) or divergence probability, a quantity which is non-zero for super-critical systems and of great relevance to the safety of nuclear systems. This will come to be useful later when we derive higher order number probability equations, as the survival probability will be an input to each of these equations and will lead to a significant simplification when calculating the derivatives and for numerical implementation.

For numerical expediency, we will assume in the rest of this paper that the physical properties of the medium (cross sections, multiplicities) are independent of time. Under these conditions, the solution is invariant to arbitrary translations of the time variable, meaning that it depends only on the time difference $s = t_f - t \geqslant 0$ and not on the initial and final times separately. Further assuming isotropic scattering and that the physical properties are spatially piecewise constant, Eq. (3) for the generating function can then be expressed in terms of the forward-time variable s as: