Annals of Nuclear Energy 103 (2017) 157–162

Contents lists available at ScienceDirect

Annals of Nuclear Energy

journal homepage: www.elsevier.com/locate/anucene

Continuous spectrum of linear transport equation for different boundary conditions

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ARTICLE INFO

Article history:
Received 19 October 2015
Received in revised form 18 August 2016
Accepted 16 January 2017

Keyword:
Continuous spectrum
Transport equation
Boundary conditions

ABSTRACT

Continuous spectrum of linear transport equation in a homogeneous slab is investigated for four different boundary conditions. These are vacuum, reflective, periodic and isotropic return conditions. It is found that the continuous spectrum is quite different for these four conditions. It is also shown that the continuous spectrum exists even if there are no infinite paths in the system.

1. Introduction

Lehner and Wing (1955) in their pioneering work showed that the time-eigenvalue spectrum (time behaviour $\sim \exp(-\lambda t)$) of linear transport operator A differs markedly from those encountered in mathematical physics. They considered one speed equation with isotropic scattering in a homgeneous slab and vacuum boundary conditions on both the surfaces.They proved that the entire half-plane $\Re(\lambda) \geqslant (v\Sigma)$ (v being the particle speed and Σ the total macroscopic cross section of the slab medium) belongs to the Continuous Spectrum ($C\sigma(A)$) of the transport operator. They also showed that no discrete eigenvalues (Point Spectrum $P\sigma(A)$) exist in this region. Further, there exist a finite number of real eigenvalues of A in the other half plane $\Re(\lambda) < (v\Sigma)$. Since then many workers have studied the spectrum of general transport operator including anisotropic scattering, energy dependence and in systems of different shapes. Thus e.g. Mika (1967) found that for energy dependent transport equation with isotropic scattering in slab geometry, there may not exist any discrete eigenvalues. All these results are summarised by Larsen and Zweifel (1974) and in the book by Kaper et al. (1982). We merely quote the results of Jörgens (1958) that for general energy dependent transport operator in a bounded system of any shape, there is no continuous spectrum if the particle speed v is bounded away from 0. If $v \to 0$ then there is a continuous spectrum in the region $\Re(\lambda) \geqslant (v\Sigma)_{min}$. Nelkin (1963) explained that the "unusual features" of Lehner and Wing (1955) spectrum are due to (i) occurrence of paths of infinite length in the system or (ii) the presence of particles of $v \to 0$ that take infinite time to cross the boundary. It is not very clear if he was referring to the occurrence of continuous spectrum, though we will call it Nelkin's conjecture hereafter.

In all these studies only vacuum boundary conditions were considered. There are few studies on other boundary conditions. Belleni-Morante (1970) considered one speed transport equation with isotropic scattering in slab geometry with periodic boundary conditions. They concluded a spectrum similar to the results of Lehner and Wing (1955). Angelescu and Protopopescu (1977) used Fourier transform method and concluded that the continuous spectrum consists only of the line $\Re(\lambda) = v\Sigma$. Finally Sahni et al. (1995) considered the case of partially reflecting boundary conditions, reflection coefficients being R_1 and R_2 (both < 1) at the left and right surfaces respectively. They found the continuous spectrum consists of an infinite set of lines, all of finite length, with their projection on $\Re(\lambda)$ axis extending over $(v\Sigma, v\Sigma - \frac{v}{4a}\ln(R_1R_2)$, where a is the half-thickness of the slab. The slopes of these lines is given by the equation $\frac{\Im(\lambda)}{\Re(\lambda)-v\Sigma} = \frac{2\pi k}{-\ln(R_1R_2)}$ where $k = -\infty, \ldots, -2, -1, 0, 1, 2, \ldots \infty$.

In this paper we focus our attention only on the continuous spectrum of transport operator in slab geometry. We consider four (i) vacuum, (ii) partially reflective, (iii) periodic and (iv) isotropic return (similar to white) boundary conditions However, our main focus will be on isotropic return boundary conditions as other three conditions have been studied previously. We record their results to highlight remarkably different spectra for these cases, to show one uniform treatment for all four conditions and briefly mention some other small additions in their treatment. We first

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http://dx.doi.org/10.1016/j.anucene.2017.01.018
0306-4549/

observe that in all previous publications the continuous spectrum was due to the part of the transport operator representing streaming and removal from the beam processes. The inscattering part of the operator hardly played any role. This fact has not been stated explicitly but is important as it allows great simplification in the mathematical treatment. We therefore drop the inscattering term and show that continuous spectrum is same as obtained earlier for three boundary conditions. We also note that this reduces the problem to essentially a one-speed problem, with particle energy occurring only as a parameter. Thus, although we study one-speed transport equation, we can infer the structure of continuous spectrum of energy dependent problem through the variation of $v\Sigma(E)$, where v is the particle speed and $\Sigma(E)$ the total macroscopic cross section of the medium.

We also want to examine Nelkin's conjecture and its modifications, if any. To this effect we consider two models. The model A is the usual transport operator with position co-ordinate $x \in [-a, a]$ and the direction cosine μ of particle direction of motion (with $x-axis$) given by $\mu \in [-1, 1]$. The model B considers the same problem in the domain $x \in [-a, a]$ and $\mu \in [-1, -v] \cup [v, 1]$, where $v > 0$ is a small arbitrary number. In the next section we introduce the problem. We then study three parts of complex λ plane, namely (a) $\Re(\lambda) < v\Sigma$, (b) $\Re(\lambda) = v\Sigma$ and (c) $\Re(\lambda) > v\Sigma$ separately in Sections 3–5. We then discuss these spectra and state our conclusions in the final section.

2. Statement of the problem

Consider the one-speed, time-dependent, general transport equation in a homogeneous slab, which is infinite along y, z directions. Thus we have

$$\frac{1}{v}\frac{\partial\Psi(x,\mu,t)}{\partial t} + \mu\frac{\partial\Psi(x,\mu,t)}{\partial x} + \Sigma\Psi(x,\mu,t) = \int_{-1}^{1}\Sigma_s(\mu' \to \mu)\Psi(x,\mu',t)d\mu' + Q(x,\mu,t) \quad (2.1)$$

Here $\Psi(x,\mu,t) = vN(x,\mu,t)$ is the angular flux (N is the number density) of the particles at the position x moving in directions with direction cosine μ at time t. The macroscopic total cross section Σ depends on the particle energy and the macroscopic scattering cross section $\Sigma_s(\mu' \to \mu)$ is a function of the indicated variables. $Q(x,\mu,t)$ is the independent source term. Assuming a time variation of the form

$$\Psi(x,\mu,t) = e^{-\lambda t}\psi(x,\mu), \quad Q(x,\mu,t) = e^{-\lambda t}q(x,\mu) \quad (2.2)$$

we get the time-independent form, namely

$$\mu\frac{\partial\psi(x,\mu)}{\partial x} + \left(\Sigma - \frac{\lambda}{v}\right)\psi(x,\mu) = \int_{-1}^{1}\Sigma_s(\mu' \to \mu)\psi(x,\mu')d\mu' + q(x,\mu) \quad (2.3)$$

We seek a solution of Eq. (2.3), $\psi \in \mathbf{L}_1$, the Banach space of absolutely summable functions i.e. $\|\psi(x,\mu)\| = \int_{-a}^{a}dx\int_{-1}^{1}d\mu\int|\psi(x,\mu)|$ is bounded ($< \infty$), for an arbitrary source $q \in \mathbf{L}_1$, with $\|q(x,\mu)\| > 0$. We note that the choice of function space $\mathbf{L}_1$ or the Hilbert space $\mathbf{H}$ (where $|\psi(x,\mu)|^2$ is summable over $-a \leqslant x \leqslant a; -1 \leqslant \mu \leqslant 1$) is arbitrary to some extent. It is introduced for mathematical convenience. Physically, the solution $\psi(x,\mu)$ of a transport problem for any given source $q(x,\mu)$ should be such that physical quantities like particle flux $\int_{\Delta\mu}d\mu\int_{\Delta x}\psi(x,\mu)dx$ over any $\Delta\mu, \Delta x$ should exist. The boundary conditions on ψ can be any one of the four conditions.

(i) **Vacuum boundary conditions**

$$\psi(-a,\mu) = \psi(a,-\mu) = 0; \quad 0 < \mu \leqslant 1 \quad (2.4a)$$

(ii) **Specularly Reflective boundary conditions** (reflection coefficients $R_1, R_2 \leqslant 1$)

$$\psi(-a,\mu) = R_1\psi(-a,-\mu); \quad \psi(a,-\mu) = R_2\psi(a,\mu); \quad 0 < \mu \leqslant 1 \quad (2.4b)$$

(iii) **Periodic boundary conditions**

$$\psi(a,\mu) = \psi(-a,\mu); \quad \psi(-a,-\mu) = \psi(a,-\mu); \quad 0 < \mu \leqslant 1 \quad (2.4c)$$

(iv) **Isotropic Return boundary conditions** ($0 < \mu \leqslant 1$)

$$\psi(-a,\mu) = 2R_1\int_0^1\mu\psi(-a,-\mu)d\mu = \alpha;$$
$$\psi(a,-\mu) = 2R_2\int_0^1\psi(a,\mu)d\mu = \gamma; \quad (2.4d)$$

where the incoming angular distributions $\psi(-a,\mu)$ and $\psi(a,-\mu)$ are independent of μ, given by two parameters α, γ. The conditions, Eq. (2.4d) imply that total partial incurrents are a fraction of the total partial outcurrents given by the reflection coefficients $R_1, R_2 \leqslant 1$.

All those complex values of λ/v for which a unique solution ψ, of Eq. (2.3), exists for every source distribution q, form the Resolvent Set of the transport operator

$$\mu\frac{\partial}{\partial x} + \Sigma - \int_{-1}^{1}\Sigma_s(\mu' \to \mu)d\mu' = \mathbf{T} - \mathbf{J} \quad (2.5)$$

The values of λ/v that are not in the Resolvent Set form the spectrum of the transport operator, Eq. (2.5), subject to the given (one of the four) boundary conditions (2.4a), (2.4b), (2.4c) or (2.4d). It is quite common to split this operator in two parts, $\mathbf{T} = \mu\frac{\partial}{\partial x} + \Sigma$ and the inscattering operator $\mathbf{J} = \int_{-1}^{1}\Sigma_s(\mu' \to \mu)d\mu'$.

As mentioned in the Introduction, this paper deals only with the continuous spectrum of the transport operator, Eq. (2.5).Our emphasis is on the effect of boundary conditions on this part of the spectrum. A closer examination of earlier papers, Lehner and Wing (1955),Larsen and Zweifel (1974) and Sahni et al. (1995) shows that the continuous spectrum is entirely due to the operator $\mathbf{T}$, which acts on both the position variable x and the direction variable μ. The Inscattering operator $\mathbf{J}$ involves only the direction variables μ, μ' and is generally a compact operator for these variables. A well known result of functional analysis states that such operators can have only eigenvalues with $\lambda = 0$ as the only point of accumulation. Indeed for isotropic scattering, $\Sigma_s(\mu' \to \mu) = \frac{\Sigma_s}{2}$, there are just two eigenvalues namely Σ_s, with unity as the corresponding eigenvector (or more generally an arbitrary function of x, independent of μ) and $\lambda = 0$ with all higher order Legendre Polynomials as eigenfunctions. There is no continuous spectrum. In all the earlier studies, continuous spectrum is obtained by constructing a set of functions $\psi_\delta(x,\mu)$ with $\|\psi_\delta\|$ always finite while $\|(\mathbf{T} - \lambda/v\mathbf{I})\psi_\delta\| \to 0$ as $\delta \to 0$ (here $\mathbf{I}$ is the identity operator) i.e. the continuous spectrum is generated by the operator $\mathbf{T}$ only. One then verifies that $\|\mathbf{J}\psi_\delta\| \to 0$ with $\delta \to 0$, thus confirming that the operator $\mathbf{T} - \mathbf{J}$ has the same continuous spectrum. Since in this paper we are primarily interested in studying only the continuous spectrum for different boundary conditions, we ignore the Inscattering operator $\mathbf{J}$ and henceforth consider the equation

$$\mu\frac{\partial\psi(x,\mu)}{\partial x} + (\Sigma - \lambda/v)\psi(x,\mu) = q(x,\mu) \quad (2.6)$$

or its compact form

$$\left[\mathbf{T} - \frac{\lambda}{v}\right][\psi(x,\mu)] = q(x,\mu) \quad (2.6a)$$