Nuclear Engineering and Technology 49 (2017) 1125–1134

Contents lists available at ScienceDirect

Nuclear Engineering and Technology

journal homepage: www.elsevier.com/locate/net

Original Article

# A multilevel in space and energy solver for multigroup diffusion eigenvalue problems

Ben C. Yee*, Brendan Kochunas, Edward W. Larsen

*Department of Nuclear Engineering and Radiological Sciences, University of Michigan, Ann Arbor, MI 48109, USA*

ARTICLE INFO

*Article history:*
Received 2 June 2017
Received in revised form
14 July 2017
Accepted 27 July 2017
Available online 5 August 2017

*Keywords:*
Multigroup diffusion
Multilevel
Eigenvalue

ABSTRACT

In this paper, we present a new multilevel in space and energy diffusion (MSED) method for solving multigroup diffusion eigenvalue problems. The MSED method can be described as a PI scheme with three additional features: (1) a grey (one-group) diffusion equation used to efficiently converge the fission source and eigenvalue, (2) a space-dependent Wielandt shift technique used to reduce the number of PIs required, and (3) a multigrid-in-space linear solver for the linear solves required by each PI step. In MSED, the convergence of the solution of the multigroup diffusion eigenvalue problem is accelerated by performing work on lower-order equations with only one group and/or coarser spatial grids. Results from several Fourier analyses and a one-dimensional test code are provided to verify the efficiency of the MSED method and to justify the incorporation of the grey diffusion equation and the multigrid linear solver. These results highlight the potential efficiency of the MSED method as a solver for multidimensional multigroup diffusion eigenvalue problems, and they serve as a proof of principle for future work. Our ultimate goal is to implement the MSED method as an efficient solver for the two-dimensional/three-dimensional coarse mesh finite difference diffusion system in the Michigan parallel characteristics transport code. The work in this paper represents a necessary step towards that goal.


## 1. Introduction

The multigroup diffusion eigenvalue problem is an approximation to the multigroup neutron transport eigenvalue problem that is widely used for reactor physics simulations. The solution is frequently used to accelerate the source iteration procedure for solving neutron transport problems via methods such as coarse mesh finite difference (CMFD) [1]. Although solving a diffusion problem requires significantly fewer computational resources than solving a transport problem, this cost is still not trivial. Many transport codes that use CMFD-like procedures (e.g., the Michigan parallel characteristics transport (MPACT) code [2]) have a CMFD eigenvalue problem with hundreds of millions of unknowns, and obtaining solutions to this problem constitutes a large portion of the computational effort.

In this work, we introduce a new multilevel in space and energy diffusion (MSED) method for solving the multigroup diffusion eigenvalue problem. This is a multicomponent method that draws from existing ideas (multigrid-in-space [3] and two-grid in energy [4]) as well as new ideas (space-dependent Wielandt shift [5]). The three primary components of MSED are: (1) a “grey” (one-group) diffusion equation, used to converge the eigenvalue and fission source, (2) a space-dependent Wielandt shift, used to reduce the number of power iterations (PIs) required for convergence, and (3) a multigrid-in-space solver, used to solve the fixed-source grey and multigroup diffusion linear systems.

The MSED method can be viewed as an extension of the CMFD method. In CMFD, the convergence of a higher-order (more unknowns) transport or nodal diffusion system is accelerated by leveraging a lower-order diffusion system. In MSED, this lower-order diffusion system is itself accelerated by simpler diffusion equations with even fewer unknowns. Fig. 1A provides a visualization of this hierarchy. Alternatively, the MSED method can be viewed as an extension of the multigrid method to nonspatial variables. Fig. 1B provides a visualization of the changes in spatial and energy grid sizes in the MSED iteration scheme. These two figures are further explained in Sections 2 and 3.

Of the methods that have been developed for reactor physics simulations, the multilevel coarse mesh rebalance (MLCMR) and multilevel surface rebalance (MLSR) methods developed by van

* Corresponding author.
*E-mail addresses:* bcyee@umich.edu (B.C. Yee), bkochuna@umich.edu (B. Kochunas), edlarsen@umich.edu (E.W. Larsen).

http://dx.doi.org/10.1016/j.net.2017.07.014

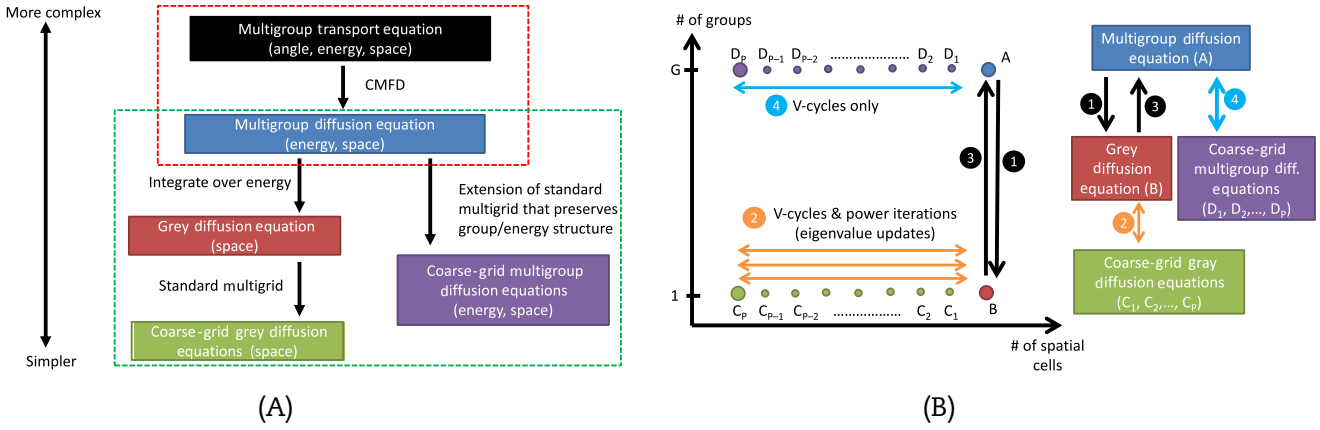


**Fig. 1.** Each figure provides an overview of the MSED iteration procedure. (A) The hierarchy of the equations in MSED is shown, with a scale on the left describing the relative complexity of these equations (i.e., the number of unknowns). The higher (red) dashed box encloses the equations used in the CMFD method, while the lower (green) dashed box encloses the equations used in the MSED method. (B) An MSED iteration is broken up into four steps, and the changes in the energy and spatial grid sizes at each step are visualized.

Geemert and others [6–8] are the most similar to MSED. The MLCMR and MLSR methods are techniques for nodal diffusion problems, which leverage solutions from a series of lower-order coarse-grid one-group diffusion equations in order to minimize the number of iterations required on the full higher-order nodal diffusion equations. In this sense, the approaches taken by the MSED method and the MLCMR/MLSR methods are similar—both methods minimize the computational effort required by shifting iterations from higher-order equations to lower-order equations that are less costly to solve. The MSED method is also similar to multilevel CMFD methods [1,9,10]. In these cited works, the multigroup CMFD equations are themselves accelerated by two-group CMFD equations.

The MSED method draws from elements of both the rebalance and multilevel CMFD concepts, but there are several important distinctions. First, the CMFD problem motivating the development of MSED is orders of magnitude larger (in terms of the number of unknowns and the number of processors used) than any of the applications in the works referenced above. Second, unlike the MLCMR/MLSR techniques, MSED operates on a diffusion/CMFD system in which the unknowns representing the neutron net current have already been eliminated—this simplifies the process of collapsing onto coarser spatial/energy grids. Third, whereas the multilevel CMFD techniques use flux-weighted cross-sections and low-order "consistency factors" (commonly denoted by $\hat{D}$, and sometimes described as a "drift" vector/term) to generate its group-collapsed equations, the MSED method uses both flux-weighted cross-sections *and* flux-weighted diffusion coefficients, and does not need additional "consistency factors" like the $\hat{D}$ in the multilevel CMFD methods. In this sense, the collapse in energy in MSED is similar to that of the MLCMR/MLSR methods. Another example in which the diffusion coefficients are group-collapsed via flux-weighting can be found in Schunert et al. [11]; there, the $S_N$ FEM-discretized transport equations are accelerated by coarse-group FEM-discretized diffusion equations. However, the coarsening in space in MSED differs from all of the aforementioned examples—the spatial variable in MSED is collapsed using a standard multigrid approach in which coarse-grid equations are error equations rather than approximations to the original system. Moreover, this collapse in MSED is performed on both the grey *and* the multigroup equations, as illustrated by Fig. 1B.

Lastly, we note that, like the MLCMR/MLSR methods, MSED uses a one-group (grey) low-order equation rather than a two-group equation. Although the referenced multilevel CMFD methods all use a two-group structure as their coarsest energy grid, we have found that the MSED method with a grey equation already performs very well. Our Fourier analysis and numerical results indicate that the MSED method, as described in this paper, has a spectral radius of ~0.1. Because of this low spectral radius and the fact that a grey system is simpler (easier to implement and solve) than a two-group system, we have not yet been compelled to study the use of a two-group system in the MSED method. Recent work by Cornejo and Anistratov, however, has shown that additional energy grid(s) between one group and $G$ groups [12,13] can provide tangible improvements in the runtime, and it may be possible to use a similar strategy to improve the MSED algorithm. In future work, we will assess the potential benefit of both introducing an extra two-group system to MSED and replacing the one-group system in MSED with a two-group system.

Another motivation for the development of the MSED method is to reduce the reliance of MPACT on "black-box" Krylov linear solvers. In recent years, many diffusion and CMFD codes have become increasingly reliant on Krylov methods for solving their linear systems. These methods are generally easy to implement due to their availability in various linear algebra libraries such as PETSc [14]. They perform reasonably well when compared to other frequently used linear solvers such as SOR or Gauss–Seidel. However, many Krylov solvers (GMRES in particular) require a significant amount of memory and may not be well-suited for high-performance computing applications where memory is a limiting resource. Moreover, Krylov methods generally do not account for the physics and structure of the problem being solved, and their convergence is typically slow for large problems unless a good physics-based or problem-dependent preconditioner is applied to the system. In many cases, such a preconditioner may not be known and, even if one exists, constructing the preconditioner and applying it to the linear system may require a significant computational effort. The approach taken by MSED is fundamentally different from those taken by the Krylov methods. Whereas Krylov methods are applicable to general linear systems, MSED is optimized only for multigroup diffusion/CMFD eigenvalue problems. MSED leverages our knowledge of the physics and structure of the multigroup diffusion problem and is designed to exploit the unique features of this problem.

In the following sections, we provide an overview of the theory for the three components of MSED, describe the full algorithm, and present results from our Fourier analysis and one-dimensional (1D) test code. This paper should be viewed as an initial report for the development of the MSED method, and the work presented in this paper is a necessary initial step towards our ultimate goal of