Original Article

The extinction probability in systems randomly varying in time

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ABSTRACT

The extinction probability of a branching process (a neutron chain in a multiplying medium) is calculated for a system randomly varying in time. The evolution of the first two moments of such a process was calculated previously by the authors in a system randomly shifting between two states of different multiplication properties. The same model is used here for the investigation of the extinction probability. It is seen that the determination of the extinction probability is significantly more complicated than that of the moments, and it can only be achieved by pure numerical methods. The numerical results indicate that for systems fluctuating between two subcritical or two supercritical states, the extinction probability behaves as expected, but for systems fluctuating between a supercritical and a subcritical state, there is a crucial and unexpected deviation from the predicted behaviour. The results bear some significance not only for neutron chains in a multiplying medium, but also for the evolution of biological populations in a time-varying environment.

1. Introduction

One interesting characteristics of a branching process is the so-called extinction probability, i.e. the asymptotic probability that when time goes to infinity, the number of entities (particles) in the system is zero.

This paper discusses some aspects of the calculation of the extinction probability in settings other than the classical case of the extinction of family trees with constant reproduction probabilities, or neutron chains in a stationary multiplying medium. The setting discussed here is the extinction probability in systems randomly varying in time. Such systems were studied before [1—3], but only the temporal evolution of the first two moments was investigated. As will be seen here, the calculation of the extinction probability is a considerably more complicated task, which necessitates the use of numerical methods.

The dependence of the extinction probability on the multiplication properties of the system in the traditional case, i.e. in a system with constant parameters (constant multiplication properties) has long been well known. For subcritical and critical systems the extinction probability equals unity, whereas for supercritical systems it is less then unity. A similar behaviour was expected also for systems with multiplication properties varying in time, with the slight difference that the definition of criticality is different (more involved) for such systems. A system is defined critical in the mean if the expectation of the neutron number converges to a constant value as time goes to infinity [1], which requires that the time-averaged reactivity of the system be negative [4,3]. Defining the value of this time-averaged subcritical reactivity as the "critical reactivity", our expectation was that the extinction probability in time-varying systems would be unity for negative average reactivities up to the critical reactivity, and less than unity for time-averaged reactivities above this value. Much to our surprise, the calculations indicated that the extinction probability remains unity even if the average reactivity is zero, in which case the system is already supercritical in the mean (the expectation of the neutron number diverges asymptotically). This is a very unexpected new result, which constitutes a crucial difference in the properties of the extinction probability for constant and temporally fluctuating systems, respectively. This result has a significance also for branching processes other than neutron multiplication, such as the population dynamics of biological systems in a time-varying environment.

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2. Theory

Ever since the classic work of Galton and Watson on the extinction of family trees [5], the extinction probability of a branching process, started by one entity (individual/particle), has always been derived from a backward type master equation. One can write down a backward master equation for the generating function $g(z, t)$ of the probability distribution $p(n, t)$,

$$g(z, t) = \sum_{n=0}^{\infty} z^n p(n, t) \tag{1}$$

of having n particles in the system at time t, given that at $t = 0$ there was one neutron in the system as [3]

$$\frac{\partial g(z, t)}{\partial t} = Q\{q[g(z, t)] - g(z, t)\} \tag{2}$$

with the initial condition

$$g(z, 0) = z. \tag{3}$$

Here, Q is the intensity of the reaction, and $q(z)$ is the generating function of the probability distribution $f(n)$ of having n particles from a reaction, i.e.

$$q(z) = \sum_{n=0}^{\infty} z^n f(n) \tag{4}$$

From this, it is immediately possible to obtain an equation for the probability $p(0, t) \equiv p_0(t)$ of extinction until time t, since $p_0(t) = g(0, t)$. The extinction probability

$$p_0 = \lim_{t \to \infty} p_0(t)$$

is obtained from (2) by assuming $dp_0(t)/dt = 0$ when $t \to \infty$, as the root of the equation

$$q(p_0) = p_0 \tag{5}$$

Actually, the above equation can be derived directly from a backward-type reasoning, considering the possible fate (=reaction) of the first individual (particle). This reasoning was given by the Dane Agner Krarup Erlang, a member of the famous Krarup family by his mother, which was about to become extinct. He published the formulation of the problem in the Danish journal Matematisk Tidsskrift in 1929 [6]. The reasoning goes as follows. The extinction probability p_0 is equal to the sum of the probabilities of the mutually exclusive events that the first particle either will not have any secondaries, with probability $f(0)$, or will have one descendant, with probability $f(1)$, which will have to die out (with probability p_0), or will have two descendants (probability $f(2)$) which both will have to die out (probability p_0^2) etc. That is,

$$p_0 = f(0) + f(1)p_0 + f(2)p_0^2 + \ldots = q(p_0) \tag{6}$$

More generally, one can also derive a similar backward type equation for the number distribution of the total number of neutrons $p(n)$ generated in the system, due to one starting neutron as

$$p(n) = \sum_{k=0}^{\infty} f(k) \prod_{n_1+n_2+\ldots n_k=n} p(n_1)p(n_2)\ldots p(n_k) \tag{7}$$

This yields for the generating function the equation

$$g(z) = q[g(z)] \tag{8}$$

from which Eq. (6) is immediately recovered by substituting $z = 0$.

Although in the above derivation time does not appear explicitly, it is clear that the equation is of a backward type. This is because the construction of the equation is based on the summing up of the probabilities of the mutually exclusive events that can happen with the starting particle on its first collision (the multiplication of the first entity/individual in the family chain). A forward equation would correspond to the summing up of the probabilities of the events of the particle(s) on their last collisions which, given the fact that all particle numbers in the system are possible, could only be given as an infinite system of coupled equations.

The above derivation is completely analogous with that of the first of the so-called Böhnel equations of nuclear safeguards [7], which specify the probability distribution of the number of neutrons leaving a multiplying sample due to one starting neutron. These are analogous to the above equations in that they do not contain time; but also in that it is not possible to derive a forward equation for any of the number distributions, for the reasons stated above. As an illustration, we quote the Böhnel equation for the probability distribution due to one starting particle, and its generating function, respectively, as [8].

$$p(n) = (1 - p)\delta_{n,1} + p \sum_{k=0}^{\infty} f(k) \prod_{n_1+n_2+\ldots n_k=n} p(n_1)p(n_2)\ldots p(n_k) \tag{9}$$

and

$$h(z) = (1 - p)z + p\, q[h(z)] \tag{10}$$

Here, p is the probability that the initial neutron will have a first collision before leaving the sample, and $h(z)$ is the generating function of $p(n)$, where the usual notation was chosen for the generating function, for easier distinction from the usual extinction problem.

This latter equation is useful to illustrate the suitability of the backward equation for the calculation of the whole probability distribution in a simple recursive manner. First, the "extinction probability" $p(0)$ of no neutrons leaving the sample is obtained in a form very similar to the traditional extinction equation as

$$p(0) = p\, q[p(0)] \tag{11}$$

This is still the same transcendental equation as for the traditional extinction probability. However, as it was shown in [8], the higher order probabilities $p(n)$, $n \geq 1$ can be obtained by solving linear algebraic equations, in which polynomial combinations of the (already known) lower order moments appear. Hence, in principle, the terms of the probability distribution $p(n)$ can be determined analytically to any arbitrary order of n.

It is thus seen that with the backward formalism, one can derive an equation directly for the extinction probability (or, for that matter, for the asymptotic number distribution of the neutrons in the system or those leaving the system), without the need of first deriving an equation for $g(z,t)$ and then substitute $z = 0$ and take the limit $t \to \infty$.

However, for systems varying randomly in time, the backward equation is not applicable. The main reason, as discussed in [1–3], is that the factorisation ansatz of the backward equation cannot be applied, because the evolution of the chains started by neutrons born simultaneously will not be independent (will be influenced simultaneously by the changing properties of the material). Hence