Ultrasonics 74 (2017) 174–185

Contents lists available at ScienceDirect

Ultrasonics

journal homepage: www.elsevier.com/locate/ultras

Nonlinear normal modes and localization in two bubble oscillators

Naohiro Sugita *, Toshihiko Sugiura

Department of Mechanical Engineering, Keio University, Yokohama 2238522, Japan

ARTICLE INFO

Article history:
Received 30 June 2016
Received in revised form 12 October 2016
Accepted 16 October 2016
Available online 18 October 2016

Keywords:
Bubble dynamics
Coupled nonlinear oscillation
Nonlinear normal modes
Nonlinear localization
Perturbation analysis

ABSTRACT

We investigated a bifurcation structure of coupled nonlinear oscillation of two spherical gas bubbles subject to a stationary sound field by means of nonlinear modal analysis. The goal of this paper is to describe an energy localization phenomenon of coupled two-bubble oscillators, resulting from symmetry-breaking bifurcation of the steady-state oscillation. Approximate asymptotic solutions of nonlinear normal modes (NNMs) and steady state oscillation are obtained based on the method of multiple scales. It is found that localized oscillation arises in a neighborhood of the localized normal modes. The analytical solutions of the amplitude and the phase shift of the steady-state oscillation are compared to numerical results and found to be in good agreement within the limit of small-amplitude oscillation. For larger amplitude oscillation, a bifurcation diagram of the localized solution as a function of the driving frequency and the separation distance between the bubbles is provided in the presence of the thermal damping. The numerical results show that the localized oscillation can occur for a fairly typical parameter range used in practical experiments and simulations in the early literatures.

1. Introduction

Small cavitation bubbles repeatedly change their volume in an oscillating pressure field, accompanied frequently with jetting and splitting into fission fragments [1] and subsequent coalescence. Such continuous response of the oscillating bubbles is referred to as stable acoustic cavitation [2], which is employed in many engineering applications such as ultrasonic cleaning [3], ultrasound imaging [4,5] and therapy [6]. The interactions between sound and cavitation bubbles have been extensively investigated since the resonant phenomenon of the bubble oscillation is an important mechanism of the above applications.

The external acoustical energy is continuously localized to oscillating bubbles and subsequently released to surroundings as the secondary radiation pressure which, in turn, drive the neighboring bubbles, leading to the mutual interaction of the oscillating bubbles. We can consider the bubbles as nonlinear oscillators coupled by the radiation pressure, and readily analyze the motion of the bubble walls on the basis of fairly mathematical treatments: that is to say the spherical bubble dynamics with time-varying radii $R_i(t)$. As the dynamical behavior of the bubble population is practically of importance to improve the validity of ultrasonic techniques, the coupled oscillation of resonant bubbles has been studied intensively for many years.

At the outset of the theoretical studies on the coupled bubble oscillation [7,8], solid mathematical consequences have been offered by linear modal analyses. Zabolotskaya [7] analyzed linear normal modes (LNMs) of two gas bubbles pulsating in a liquid based on the Lagrangian formalism, and showed that the linear normal frequencies depends on the separation distance between the two bubbles. Takahira et al. [8] provided a general derivation of coupled N bubble dynamics accounting for the translational motion and deformation of the bubbles on a basis of a potential solution. The resulting eigenvalue problem concluded that the eigenfrequency of the fundamental normal mode is much smaller than that of an unbounded single bubble. However, in contrast to linear systems, extremely complex behaviors are encountered in nonlinear systems [9]. Although nonlinear spherical dynamics of a single bubble and its bifurcation structures such as subharmonic generation, period-doubling bifurcation and chaotic oscillation have been explored [10], little is known about the bifurcation structures of the coupled bubble dynamics; most of the studies [11–15] have been employed numerical techniques.

The numerical study of Takahira et al. [11] demonstrated the period-doubling bifurcation and accompanying chaotic oscillation of interacting multi-bubble systems. The fundamental feature identified in the analysis is that equal-sized bubbles with the same initial radii arranged in a symmetrical configuration all take on the same behavior similar to that of an unbounded bubble, whereas bubbles in a cluster with different initial equilibrium radii cannot oscillate independently from one another but experience a

* Corresponding author.
E-mail address: sugita@keio.jp (N. Sugita).

http://dx.doi.org/10.1016/j.ultras.2016.10.008

collective behavior. Macdonald [12] also reported numerical results of the same collective behavior in the multi-bubble interaction of ultrasound contrast agent microbubbles.

Herein, we particularly focus on nonlinear localization [9,16,17] or symmetry-breaking bifurcation [18] of the mutual bubble interaction in which the total vibrational energy of the system is confined to some bubbles due to the nonlinearity of the bubble oscillation even though they are equally-sized and arranged in a symmetric configuration. Similar nonlinear phenomenon has been theoretically investigated as spatial resonance in a damped and periodically driven chain and oscillator arrays with a periodic boundary condition [19,20], and experimentally observed in micromechanical systems [21,22]. This symmetry-breaking property is one of the distinctive feature of the localized oscillation considered in this study. The assumption of the symmetrical arrangements and equal-sized assumption have been used in numerical investigation of the effects of bubble sizes and spatial arrangement on the coupled bubble dynamics [13–15,23]. However, the fundamental bifurcation structure of the coupled bubble dynamics has not been addressed because most of the above studies are based on numerical investigation.

The linear modal analyses have been definitely powerful tools for interpreting the underlying linear system. However, they are still inadequate to properly describe the complicated nonlinear phenomena. For a general survey of the bifurcation structure of the coupled bubble dynamics, analytical investigation of nonlinear normal modes (NNMs) [9,16,24] is an essential approach to a greater insight on the structural nature of the multi-bubble dynamics. At the first attempt of NNMs, Rosenberg [25–27] extended straightforwardly the concept of LNMs to nonlinear vibration systems and defined an NNM as a vibration *in unison* where all mass points in the system display periodic motions with the same period. In the definition, all displacements pass through their equilibrium points and reach their extreme values simultaneously. It should be also noted that NNMs inherit the invariance property of LNMs (i.e., motions that depart from the NNM confined in it for all time), which is exploited to derive the NNMs in the perturbation analysis of this study.

There have been a few studies which used a perturbative method to obtain the steady-state solution of bubble oscillation. Prosperetti [28] presented a second order steady-state solution of Rayleigh's equation of motion for the bubble wall by means of an asymptotic expansion method. The analytical result enabled it evident to predict the multivalued solution of the nonlinear oscillation and the unstable region of subharmonic resonance as well as their hysteresis behavior. Francescutto [29] used an asymptotic method of multiple scales to obtained explicit and simpler formulas for the second order approximate solution. Nevertheless, nonlinear resonance of the vibration modes among multiple bubbles are still unclear since these results are for a single bubble. We employ the method of multiple scales [24,30] to derive NNMs of the coupled bubble oscillation and investigate the internal resonance [24] of the steady-state amplitude and the phase shift.

In the present study, we will restrict the analysis to a resonant pair of two bubbles. In order not to limit the generality, the bubble sizes are allowed to be different in the perturbation analysis (Section 3), but assumed to be similar so that the two uncoupled natural frequencies of isolated bubbles have a slight difference by the order of $O(\epsilon^2)$ where ϵ is a dimensionless oscillation amplitude. Since the aim of this paper is to investigate the bifurcation structure of the radial dynamics of a resonant pair of bubbles, the separation distance of the bubbles is assumed to be unchanged by the translational instability [31] due to Bjerknes forces, while it is important to account for the transient response and hysteresis property for a full understanding of the bubble structure dynamics [32]. The circumstances of a fixed bubble distance is not improbable but achieved in the case of surface cavitation bubbles attached on a solid surface [33,34]. Because of the adhesion between the bubble and wall surface the bubble mobility is decreased, and the bubble distances tends to remain almost fixed. Additionally, the effect of the wall boundary is replaced with a mirror image of the real bubble. This allows the dynamics of the hemispherical bubble to be well described by the Rayleigh–Plesset equation for a spherical bubble in an unbounded space.

In Section 2, the equations of radial motion for the coupled dynamics of two spherical gas bubbles are presented, and the linear theory is summarized to illustrate the basic concept of normal modes. Perturbation analysis in Section 3 provides the approximate solution of the steady-state oscillation, and the NNMs of the two-bubble system are developed from the perturbation solution obtained. Section 4 performs numerical calculations for large amplitude oscillation in a broader range of parameter space. Important findings and conclusions are summarized in Section 5.

2. Radial dynamics of two spherical bubbles

We consider two gas bubbles separated by a fixed distance in a liquid driven by a stationary sound field sketched in Fig. 1. The wave length is assumed larger enough for the two bubbles to experience the equal driving pressure. Bubble oscillations are inertially controlled by periodic pressure change in the far field, and develop a secondary sound field without distorting each others' sphericity. The radiation pressure induced by one of the bubbles, bubble 1, measured at the center of the other bubble, bubble 2, is

$$p_r(d,t) = \frac{\rho}{4\pi d}\frac{d^2V_1(w)}{dt^2}\bigg|_{w=t-\frac{d}{c}} \tag{1}$$

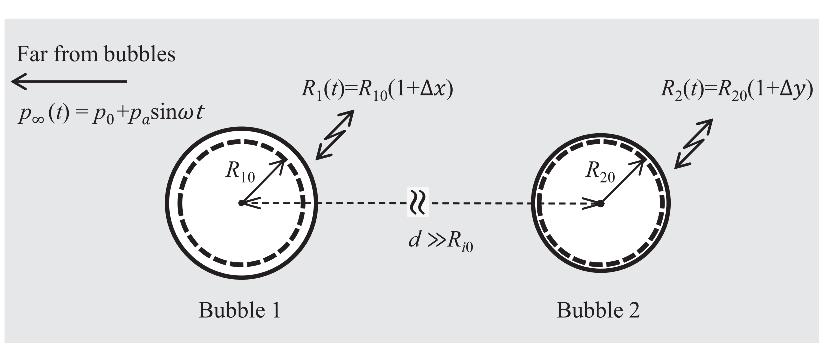

Fig. 1. Schematic of two oscillating bubbles.