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Scattering of Airy elastic sheets by a cylindrical cavity in a solid

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ABSTRACT

The prediction of the elastic scattering by voids (and cracks) in materials is an important process in structural health monitoring, phononic crystals, metamaterials and non-destructive evaluation/imaging to name a few examples. Earlier analytical theories and numerical computations considered the elastic scattering by voids in plane waves of infinite extent. However, current research suggesting the use of (limited-diffracting, accelerating and self-healing) Airy acoustical-sheet beams for non-destructive evaluation or imaging applications in elastic solids requires the development of an improved analytical formalism to predict the scattering efficiency used as *a priori* information in quantitative material characterization. Based on the definition of the time-averaged scattered power flow density, an analytical expression for the scattering efficiency of a cylindrical empty cavity (i.e., void) encased in an elastic medium is derived for compressional and normally-polarized shear-wave Airy beams. The multipole expansion method using cylindrical wave functions is utilized. Numerical computations for the scattering energy efficiency factors for compressional and shear waves illustrate the analysis with particular emphasis on the Airy beam parameters and the non-dimensional frequency, for various elastic materials surrounding the cavity. The ratio of the compressional to the shear wave speed stimulates the generation of elastic resonances, which are manifested as a series of peaks in the scattering efficiency plots. The present analysis provides an improved method for the computations of the scattering energy efficiency factors using compressional and shear-wave Airy beams in elastic materials as opposed to plane waves of infinite extent.

1. Introduction

The spectral analysis by modeling of the acoustic scattering from voids and microcracks in materials [1] is an active area of research in NDT&E applications for the estimation of the failure prediction threshold in structural health monitoring. Applications in geophysics and biomedical acoustics, where the characterization of the medium around an inclusion and the inverse estimation of the elastic parameters from scattering data provide assistance in the diagnosis, would benefit from such numerical predictions. Such computations are also relevant in emergent areas where the elastic scattering data from fiber-reinforced materials [2] (used extensively in the industry) and periodic structures such as metamaterials [3] are correlated with the results of analytical modeling.

In this process, the energy of the incident waves is scattered from the embedded inclusion and its surrounding medium. The description of this effect is formulated based on the extinction theorem (devised originally from the field of optics [4]), known also as the "cross-section theorem" [5], where the scattering efficiency (or cross-section [6,7]) of a target can be evaluated stemming from the law of the conservation of energy. Thus, a useful methodology for characterizing the scatterer can be developed based on calculating the energy flux scattered by the inclusion (or void), and compare it with that of the incident waves.

Standard methods have used the multipole expansion in cylindrical wave functions in cylindrical coordinates [8–13]. Nevertheless, such analyses have been mainly restricted to the case of plane waves, where the incident wavefronts (i.e. iso-surfaces of constant phase) are infinite parallel planes of constant amplitude normal to the direction of wave propagation. In practical applications, however, a finite *beam* bounded in space is utilized. Recent work using nonparaxial Gaussian "acoustical sheets" [14] (i.e. finite beams in 2D) has been suggested for particle manipulation in fluids. Nonetheless, the formalism is not applicable to a particle embedded in an elastic matrix, because of the elastodynamic coupling with the host medium that affects and alters the scattering from the particle [15,16]. Acoustical sheets consist of a thin "slice" of a confined beam where the field is distributed in the 2D cross-sectional plane [17], with invariance along the z-direction (Fig. 1). Thus, it is of some importance to extend the previous

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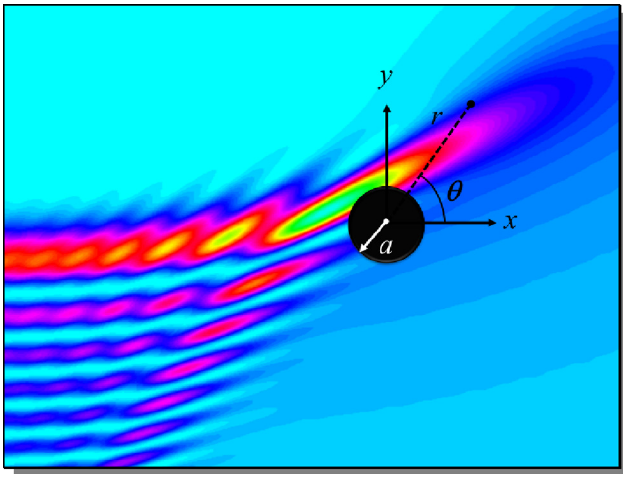

Fig. 1. Graphical representation of an Airy elastic-sheet beam incident upon a circular cylindrical cavity of radius *a* embedded in an elastic matrix. The beam propagates with an arbitrary direction with respect to the center of the cavity with invariance along the *z*-axis (perpendicular to the plane of the figure).

methods and devise an improved formalism applicable to "sheets" in elastic materials, which are defined here as "elastic-sheets".

In this work, in contrast to plane waves of infinite extent, nonparaxial Airy elastic-sheet beams are suggested and examined from the standpoint of elastic scattering theory by a circular cylindrical void encased in an elastic matrix. The scattering energy efficiency is determined and evaluated numerically with particular emphasis on the ratio of the compressional to the shear wave speeds, which determine the property of the surrounding elastic matrix material. The computations, which suggest the use of Airy elastic sheet beams as potential candidates in contrast to plane waves, may be relevant in various applied acoustics areas involving NDE&T, biomedical imaging and geophysics applications to name a few examples. Notice that Airy beams resist diffraction [18] and have the ability to reform [19] after encountering a small obstacle, as long as the whole beam is not blocked. Furthermore, the wave propagation of nonparaxial Airy beams follows a parabolic (nonlinear) trajectory curving through space [20], offering the possibility of imaging around corners for the detection of defects and cracks in materials.

These features of Airy beams provide the impetus here to investigate the scattering from an empty cavity (i.e. void) embedded in an elastic matrix, encompassing the cases of incident compressional (c) and shear (s) wave incidences, although a recent work investigated the extinction and absorption efficiencies from a fluid-filled viscous inclusion [21]. Such an analysis for the scattering of Airy elastic sheets by a cylindrical void seems to be nonexistent yet.

In this analysis, the formalism for the scattering efficiency is derived based upon the multipole expansion method using cylindrical wave functions, and closed-form partial-wave series expansions (known also as generalized Rayleigh series) in cylindrical coordinates. Stemming from the equations of elastodynamics, and the integration of the time-averaged scattered power flow density [22] using the far-field scattering, the corresponding scattering efficiencies (or cross-sections) for both the c and s waves are obtained. Numerical computations illustrate the analysis for a 2D Airy beam with arbitrary incidence and particular emphasis is given on the beam parameters and shift from the center of the circular cylindrical inclusion. These effects predicted in 2D will constitute the basis for future research dealing with the 3D case, and this analysis should assist along that direction of research.

2. Method

2.1. Elastic wave scattering

The analysis is started from the basic equation of motion for the particle displacement vector in an elastic medium $\mathbf{u}$ expressed as [23,24],

$$(\lambda + 2\mu)\nabla(\nabla \cdot \mathbf{u}) - \mu \nabla \times (\nabla \times \mathbf{u}) = \rho \partial^2 \mathbf{u}/\partial t^2, \quad (1)$$

where λ and μ are the Lamé coefficients of the homogeneous isotropic elastic matrix medium and ρ its density.

The displacement vector $\mathbf{u}$ is expressed as the sum of the gradient of a scalar potential Φ and the curl of a solenoidal vector potential $\mathbf{\Psi}$ (satisfying the gauge invariance condition $\nabla \cdot \mathbf{\Psi} = 0$) as,

$$\mathbf{u} = \nabla\Phi + \nabla \times \mathbf{\Psi}. \quad (2)$$

The displacement equations are satisfied if the potentials Φ and $\mathbf{\Psi}$ satisfy the Helmholtz equations for the solid medium,

$$(\nabla^2 + k_c^2)\Phi = 0, \quad (3)$$

$$(\nabla^2 + k_s^2)\mathbf{\Psi} = \mathbf{0}, \quad (4)$$

where $k_c = \omega/c_c = \omega/\sqrt{(\lambda + 2\mu)/\rho}$, and $k_s = \omega/c_s = \omega/\sqrt{\mu/\rho}$, refer to the longitudinal and transverse wave numbers in the elastic matrix, respectively.

It follows from symmetry considerations that the vector potential $\mathbf{\Psi}(\psi_r = 0, \psi_\theta = 0, \psi_z \neq 0)$ has only one nonzero component along the z-direction [25].

2.2. Compressional Airy beam incidence

Consider a monochromatic Airy beam composed of compressional waves propagating in an elastic medium with arbitrary incidence with respect to the center of a void cylindrical inclusion (Fig. 1). Using the separation of variables (non-singular) solution of the Helmholtz equation, the incident displacement potential field for the compressional wave is expressed in cylindrical coordinates as [26–28],

$$\Phi_{inc}^{(c)}(r, \theta) = \phi_0 \sum_{n=-\infty}^{+\infty} b_n^{(c)} J_n(k_c r) e^{in\theta}, \quad (5)$$

where ϕ_0 is the amplitude, $b_n^{(c)}$ are the beam-shape coefficients (BSCs) that characterize the incident Airy beam, and $J_n(\cdot)$ is the cylindrical Bessel function of the first kind. A time-harmonic variation in the form of $e^{-i\omega t}$ is assumed, but suppressed for convenience from Eq. (5) (and the subsequent ones) since the space-dependent field is only considered. Based upon the analysis introduced in the context of electromagnetic theory [29] and extended in the context of particle manipulation using Airy acoustical sheets in fluids [17], the BSCs corresponding to the elastic compressional beam can be expressed as,

$$b_n^{(c)}\Big|_{Airy} = i^n \left(\frac{k_c y}{2\pi}\right) \times \int_{-1}^{+1} e^{(\alpha - ik_c y q)^3/3} e^{ik_c(x_0\sqrt{1-q^2} + y_0 q)} e^{-in \sin^{-1}(q)} dq, \quad (6)$$

where $k_c y$ is a non-dimensional transverse scaling factor, and $\alpha > 0$ is a non-dimensional parameter that determines the strength of the incident field [17,29]. The variables x_0 and y_0 are the coordinates of the Airy beam in the transverse plane, such that the point (x_0, y_0) = (0, 0) corresponds to the center of the beam. Eq. (6) can be evaluated using a single standard numerical integration procedure.