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A generalization of the analytical least-squares solution to the 3D symmetric Helmert coordinate transformation problem with an approximate error analysis

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Abstract

The symmetric Helmert transformation model is widely used in geospatial science and engineering. Using an analytical least-squares solution to the problem, a simple and approximate error analysis is developed. This error analysis follows the Pope procedure solving nonlinear problems, but no iteration is needed here. It is simple because it is not based on the direct and cumbersome error analysis of every single process involved in the analytical solution. It is approximate because it is valid only in the first-order approximation sense, or in other words, the error analysis is performed approximately on the tangent hyperplane at the estimates instead of the original nonlinear manifold of the observables. Though simple and approximate, this error analysis's consistency is not sacrificed as can be validated by Monte Carlo experiments. So the practically important variance-covariance matrix, as a consistent accuracy measure of the parameter estimate, is provided by the developed error analysis. Further, the developed theory can be easily generalized to other cases with more general assumptions about the measurement errors.

Keywords: Symmetric Helmert coordinate transformation; Least-squares; Error analysis; Consistency; Monte Carlo experiments

1. Introduction

Coordinate transformation between reference frames is common in geospatial science and engineering (Lu et al., 2014). The functional model of this problem, i.e., the transformation model, is essentially complex because of the movements of the crustal plates (Yang and Zeng, 2009; Yang et al., 2011) and/or the distortions of the networks that realize the frames (Grgic et al., 2016; Leick and van Gelder, 1975). Due to the practical complexity, there can hardly be a true functional model connecting two different frames, and hence, the aim of the modeling is to select or determine a model that is practically useful rather than theoretically true (Burnham and Anderson, 2002). Though a model can be selected or validated from a set of candidates using the measurements by cross validation (Chiu and Shih, 2014), hypothesis test (Lehmann, 2014), or the Akaike information criterion (Felus and Felus, 2009), it is common practice in surveying that a model is pre-assumed and only the parameters in the model is estimated. Among the many, the Helmert model is widely adopted. Three different transformations are involved in the Helmert model, i.e., the translation, the rotation and the isotropic scale. However in geodesy, the Helmert model is

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theoretically feasible. This is because the transformation is not a physical one but introduced artificially in obtaining the mathematical coordinates by solving a rank-deficient problem with different constraints (Dermanis, 1994; Sillard and Boucher, 2001). In combining or stacking different solutions by different analysis centers or at different instances, Helmert transformations are estimated maybe as nuisance parameters. Note that the reduced or expanded (time variant model), rather than the standard Helmert model may be involved (Chatzinikos and Dermanis, 2016). It is natural to formulate the translation and the scale as a 3×1 real vector and a positive scalar, respectively. However, the case for the rotation is a little more complex. In the literature of the datum transformation, different parameters have been employed to represent the rotation, for instances, the direction cosine matrix (DCM) (Chang, 2015; Grafarend and Awange, 2003), the quaternion (Sanso, 1973; Shen et al., 2006), the Euler angles (Fang, 2015; Yang, 1999), etc. Besides these mentioned, there are many others that can be used to formulate a 3D rotation, such as the rotation vectors, the Gibbs vectors or the Rodrigues parameters, the modified Rodrigues parameters, the Cayley-Klein parameters (Q. Wang et al., 2016). For small angles, those different representations have equivalent numerical properties because all can degenerate into the same 3×1 vector (Li et al., 2012; Yang, 1999). However, for rotation with angles that cannot be practically small, representations with redundant parameters, such as the DCM and the quaternion, should be preferred to those with minimum parameters, because they can avoid singularities and strong nonlinearities (Q. Wang et al., 2016). The Helmert transformation problem is then defined as estimating the parameters using measured coordinates in the two involved frames. As can be seen in the sequel, the error analysis does not depend on the specific representation used in the solution, however, for simplicity and without loss of generality, the DCM is employed.

Representation of the estimation errors of the parameters is also an issue needing some explanation. It is natural to use plain additive errors for the scale and the translation parameters. Additive errors can also be adopted for rotations with small angles. However for rotations with practically non-small angles, it is not so simple. For redundant formulations, e.g., the DCM and the quaternion, constraints exist for the additive errors because both the estimated and the true parameters have constraints. Even for minimum formulations, e.g., the Euler angles, the rotation vector, the Gibbs vector, and the modified Rodrigues parameters, additive errors are not convenient to be manifested in the error analysis. Alternatively, a 3×1 error vector is defined according to the multiplicative error of the rotation. This error vector holds for small estimation errors. Note here it is the estimation error of the rotation angle rather than the rotation angle itself that is assumed small. Apparently, this assumption can hold in general for a relatively good estimation method. Further this error vector applies to any of the representations of the rotation, or in other words, for the same rotation and the same estimate, the error vectors for different representations equals to each other. As the error vector is derived from the multiplicative rotation error, it is also called multiplicative error vector for brevity. Note that for different representations, the "multiplication" has different forms. For more information concerning representation of the rotation parameter estimation errors, see (Q. Wang et al., 2016).

Besides the functional model, i.e., the Helmert model, a stochastic model should be employed to clearly describe the statistical properties of the measurement errors. This is apparently important, because no statistically meaningful solution can be derived without it, though some algebraic and geometric solutions can be obtained (Awange and Paláncz, 2016). First of all, the stochastic model should tell where the measurement errors should be introduced. It seems to be a simple question: where the measurements are involved, the measurement errors should be introduced. However, in many works in the literature, parts of the errors are neglected, though implicitly in some cases. To be more specific, coordinate measurements in only one frame were considered noisy sometimes. This neglecting, or approximation to be more objective, has practical reasons, because it can result in a standard Gauss-Markov model which can be readily solved by, say, the least-squares method (Koch, 1999; Teunissen, 2000). Apparently, in order to be more rigorous, an alternative stochastic model should be employed in which measurements in both frames should be assumed with errors. This model has already been widely investigated in different disciplines and with different names, for instances, the Gauss-Helmert model (Koch, 2014), the error-in-variables model (Xu and Liu, 2014), the measurement error model (Carroll et al., 2006), the mixed model (Leick et al., 2015). The least-squares solution under this model is also called a total least-squares solution (Fang, 2013; Schaffrin and Felus, 2008; Schaffrin and Wieser, 2008). However the total least-squares is not a new adjustment method but exactly the least-squares adjustment for, say the Gauss-Helmert model (Neitzel, 2010; Schaffrin, 2006; Teunissen, 1985; Xu et al., 2012). Also, in the framework of the traditional least-squares theory, using the Pope iteration procedure, the problem can be readily solved, and furthermore with an efficient estimate of the variance-covariance matrix (VCM) for the parameter estimates, for details see (Pope, 1972), for the use in the specific Helmert transformation problem see (Dermanis, 2015). The Helmert coordinate transformation with the Gauss-Helmert stochastic model is also called the symmetric Helmert transformation (Felus and Burtch, 2009; Teunissen, 1988). This terminology is followed in this work for brevity. Of course, a complete stochastic model should also tell the detailed statistical properties of the measurement error, i.e., its distribution. Gaussian distribution is assumed here, robustness against this assumption is not tackled in this work, interested readers is referred to (Yang, 1999).

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