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An analytical approach to the CMB anisotropies in a spatially closed background

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ABSTRACT

The scalar mode temperature fluctuations of the cosmic microwave background has been derived in a spatially closed universe from two different methods. First, by following the photon trajectory after the last scattering and then from the Boltzmann equation in a closed background and the line of sight integral method. An *analytic* expression for the temperature multipole coefficient has been extracted at the hydrodynamical limit, where we have considered some tolerable approximations. By considering a realistic set of cosmological parameters taken from a fit to data from Planck, the TT power spectrum in the scalar mode for the closed universe has been compared with numerical one by using the CAMB code and also latest observational data. The analytic result agrees with the numerical one on almost all scales. The peak positions are in very good agreement with numerical result while the peak heights agree with that to within 10% due to the approximations have been considered for this derivation.

1. Introduction

The cosmological parameters of the standard big bang model can be determined or considerably constrained by comparing the predictions of theoretical cosmological models with the data on the CMB by observation, such as WMAP and Planck. The theoretical derivation of the spectra of the CMB temperature anisotropies and polarizations has been archived by sophisticated numerical calculation codes such as CAMB [1–3] and CMBFAST [4] that give the spectra $C_{XX',\ell}$ which involves several cosmological parameters. However, analytical studies give us a great insight into the problem for understanding how various underlying physical effects give rise to specific observational behavior. In particular, the analytical studies are helpful in revealing the explicit dependencies of the CMB spectra on cosmological parameters and possible degeneracies between them.

There are several works in the field that extracted an analytical expression for the $C_{XX',\ell}$ in a flat universe. In Refs. [5–11] you can find all analytical spectra by considering the tensor perturbation as a source. Refs. [12–15] gave the analytic calculation of the scalar mode temperature power spectrum in Newtonian gauge while Refs. [16,17] gave the scalar mode analytic power spectra in synchronous gauge. Ref. [18] also gave a unifying framework for all spectra in both tensor and scalar modes. However, the analysis is still incomplete by lacking analytic expressions for the closed and open geometry. viewing these, we are going to perform a detailed analytic calculation of scalar mode (in synchronous gauge) temperature power spectrum $C^S_{TT,\ell}$ in a spatially closed background. We will apply some of the results and techniques developed in the study of the cosmic microwave background anisotropies in a flat spatial geometry to the closed case and compare the consequences with that of numerical calculation and latest observational data. By applying a series of tolerable approximations that lead to a simple *analytic* formula for the CMB power spectrum, we provide transparent information about the dependencies of the CMB spectra on cosmological parameters. We extract an analytic formula for the closed universe temperature fluctuations by imposing the effect of curvature into the Boltzmann equation for the photons and using the line of sight method without using any recursion relations which used by others for numerical calculations. We derive this expression from a more geometrical approach, by following the photon trajectory from the last scattering surface until now in a spatially closed background. We also calculate the multipole coefficient *analytically*, in hydrodynamic limit and compare the result with those of a flat universe and observational data.

In the following section, we give a brief overview of the perturbation theory in a spatially closed universe and its applications in the present paper. In Section 3, we extract the temperature fluctuations by following the photon trajectory from the last scattering surface in the spatially closed background. In Section 4, we introduce the Boltzmann equation for the photons in the spatially

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closed background and extract the temperature fluctuations by using the line of sight integral method. The approach presented here for temperature fluctuations can also be used for extracting the polarization multipoles $C^S_{TE,\ell}$ and $C^S_{EE,\ell}$ from the Boltzmann equation. In Section 5, at first, we introduce a general formula for the temperature multipole coefficient in a closed background and then by considering that the evolution of cosmological perturbations is primarily hydrodynamics, among some other appropriate approximations, we extract an analytic formula for the temperature power spectrum $C^S_{TT,\ell}$. In Section 6, we plot the TT power spectrum curve extracted in Section 5 using a realistic set of cosmological parameters and compare it with the numerical one by CAMB and also the curve from latest observational data (Planck 2015). Several interesting properties of CMB anisotropies are revealed in analytic expression along with the power spectrum dependence on cosmological parameters. We conclude the article by a brief review that remarks the main outcomes of this paper.

2. The perturbation theory in a spatially closed background; a short review

The theory of the linear perturbations is an important part of the modern cosmology which explains CMB anisotropies and the origin of structure formation. There is enough references for this theory in a spatially flat universe and has been investigated for a spatially closed universe recently [19].

The perturbed metric is:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} \tag{1}$$

where $\bar{g}_{\mu\nu}$ and $h_{\mu\nu}$ are the unperturbed metric and the first order perturbation, respectively. Note that $\bar{g}_{\mu\nu}$ is the FLRW metric which in the comoving spherical polar coordinates can be written as

$$\bar{g}_{00} = -1$$
$$\bar{g}_{rr} = \frac{a^2(t)}{1-Kr^2} \qquad \bar{g}_{\theta\theta} = a^2(t)r^2 \qquad \bar{g}_{\varphi\varphi} = a^2(t)r^2\sin^2\theta$$

Perturbation in the metric leads to perturbation in the Ricci and energy-momentum tensor. We can decompose the metric perturbation and energy-momentum tensors into the scalar, vector and tensor modes from their transformation properties under spatial rotations and derive the field equations accordingly [19].

Decomposition into the scalar, vector and tensor modes of the metric perturbation and energy-momentum tensor would be as follows:

$$\begin{aligned}
h_{00} &= -E \\
h_{i0} &= a\,(\nabla_i F + G_i) \\
h_{ij} &= a^2\,(A\,\tilde{g}_{ij} + H_{ij}B + \nabla_i C_j + \nabla_j C_i + D_{ij}) \\
\delta T_{00} &= -\bar{\rho}\,h_{00} + \delta\rho \\
\delta T_{i0} &= \bar{p}\,h_{i0} - (\bar{\rho}+\bar{p})(\nabla_i \delta u + \delta u_i^V) \\
\delta T_{ij} &= \bar{p}\,h_{ij} + a^2(\tilde{g}_{ij}\delta p + H_{ij}\Pi^S + \nabla_i \Pi_j^V + \nabla_j \Pi_i^V + \Pi_{ij}^T).
\end{aligned}$$

where ∇_i is the covariant derivative with respect to the spatial unperturbed metric $\tilde{g}_{ij}(= a^{-2}\bar{g}_{ij})$ and $H_{ij} = \nabla_i\nabla_j$ is the *covariant Hessian operator*. All the perturbations A, B, E, F, C_i, G_i and D_{ij} are functions of $\mathbf{x}$ and t which satisfy

$$\nabla^i C_i = \nabla^i G_i = 0$$
$$\tilde{g}^{ij} D_{ij} = 0 \quad \nabla^i D_{ij} = 0 \quad D_{ij} = D_{ji}$$

On the other hand, all above perturbative quantities have been considered as random fields on $S^3(\alpha)$ (a 3-sphere of radius α), because they are defined on a homogeneous and isotropic space [20,21]. So they can be described by their Fourier transformation. There are many different Fourier transform convention, however here we are going to expand each mode of the perturbation fields in terms of the corresponding eigenfunctions of the *Laplace–Beltrami* operator. This operator reduces to the ordinary Laplacian in a flat background. In pseudo-spherical coordinates with the line element

$$ds^2 = \alpha^2\,(d\chi^2 + \sin^2\chi\, d\theta^2 + \sin^2\chi\,\sin^2\theta\, d\varphi^2) \tag{2}$$

one gets the following eigenvalues and eigenfunctions for the Laplace–Beltrami operator:

$$\begin{aligned}
\nabla^2\Phi &= -k_n^2\Phi \qquad \nabla^2 = \tilde{g}_{ij}H_{ij} = \tilde{g}_{ij}\nabla_i\nabla_j \\
\Phi &= \mathcal{Y}_{n\ell m}(\chi,\theta,\varphi) = \Pi_{n\ell}(\chi)\,Y_{\ell m}(\theta,\varphi) \\
k_n^2 &= \frac{n^2-1}{\alpha^2} \qquad n = 1, 2, \cdots
\end{aligned}$$

where $\Pi_{n\ell}(\chi)$ is the hyperspherical Bessel function satisfying the following equation

$$\frac{d^2\Pi_{n\ell}(\chi)}{d\chi^2} + 2\cot\chi\,\frac{d\Pi_{n\ell}(\chi)}{d\chi} + \left[(n^2-1) - \frac{\ell(\ell+1)}{\sin^2\chi}\right]\Pi_{n\ell}(\chi) = 0 \tag{3}$$

In a flat background, the hyperspherical Bessel function reduces to the ordinary spherical Bessel function $j_\ell(\nu\chi)$. Also we introduce the generalized wave number in closed space q_n as

$$q_n = \sqrt{k_n^2 + \frac{1}{\alpha^2}} = \frac{n}{\alpha}$$

We can expand the scalar perturbative quantity $A(\mathbf{x}, t)$ in terms of *Laplace–Beltrami* operator eigenfunctions as below:

$$A(\mathbf{x}, t) = \sum_{n\ell m} A_{n\ell m}(t)\,\mathcal{Y}_{n\ell m}(\chi,\theta,\varphi) \tag{4}$$

This is the initial conditions that depend on the direction, not the perturbation itself, so a perturbation can be shown by a time-dependent normal mode $A_n(t)$ with an overall normalization factor α_{lm}. $A_{n\ell m}(t)$ just like $A(\mathbf{x}, t)$ is a scalar random field and one of the simplest statistics for it is the two-point covariant function denoted by $\langle A_{n\ell m}A^*_{n'\ell'm'}\rangle$. Here $\langle\ \ \rangle$ means the ensemble average which equals the spatial average according to the *ergodic* theorem. The homogeneity and isotropy imply that

$$\langle\alpha_{\ell m}\,\alpha^*_{\ell' m'}\rangle = \delta_{\ell\ell'}\,\delta_{mm'} \quad \langle A_n(t)\,A^*_{n'}(t)\rangle = A_n^2(t)\,\delta_{nn'}$$

so the two-point covariant function of $A_{n\ell m}(t)$ is

$$\begin{aligned}
\langle A_{n\ell m}A^*_{n'\ell'm'}\rangle &= \langle\alpha_{\ell m}\,A_n(t)\,\alpha^*_{\ell m}\,A^*_{n'}(t)\rangle \\
&= \langle\alpha_{\ell m}\,\alpha^*_{\ell'm'}\rangle\langle A_n(t)\,A^*_{n'}(t)\rangle \\
&= A_n^2(t)\,\delta_{nn'}\,\delta_{\ell\ell'}\,\delta_{mm'}
\end{aligned}$$

and for any scalar random field A, we will have

$$A(\mathbf{x}, t) = \sum_{n\ell m}\alpha_{\ell m}\,A_n(t)\,\mathcal{Y}_{n\ell m}(\chi,\theta,\varphi) \tag{5}$$

Also, we can decompose an arbitrary tensor random field using the scalar eigenvalues of the Laplace–Beltrami operator and their covariant derivatives as follows

$$A^{ij}(\mathbf{x}, t) = \sum_{n\ell m}\alpha_{\ell m}\left[\frac{1}{3}A_{nT}(t)\,\tilde{g}^{ij}\,\mathcal{Y}_{nlm}(\chi,\theta,\varphi) + A_{nTL}(t)\left(k_n^{-2}H^{ij}\mathcal{Y}_{nlm}(\chi,\theta,\varphi)\right)\right], \tag{6}$$

where $A_{nT}(t)$ and $A_{nTL}(t)$ are the trace and traceless parts of the tensor A^{ij} respectively [22,23].