Astroparticle Physics 93 (2017) 1–7

Contents lists available at ScienceDirect

Astroparticle Physics

journal homepage: www.elsevier.com/locate/astropartphys

Estimating significance in observations of variable and transient gamma-ray sources

Ori M. Weiner[a,*], T.B. Humensky[a], Reshmi Mukherjee[b], Marcos Santander[b]

[a] Physics Department, Columbia University, New York, NY, 10027, USA
[b] Department of Physics and Astronomy, Barnard College, Columbia University, New York, NY 10027, USA

ARTICLE INFO

Article history:
Received 16 February 2017
Revised 2 May 2017
Accepted 19 May 2017
Available online 20 May 2017

Keywords:
Gamma-ray astronomy
Statistical methods
Gamma-ray bursts
Transient sources

ABSTRACT

The standard method for estimating the statistical significance of a gamma-ray source detection is that introduced by Li & Ma (1983), Eq. (17). In observing sources with time-dependent light curves, one can improve on this method by including approximate *a priori* knowledge of the source temporal behavior. A maximum-likelihood-based approach is suggested that provides an improvement in sensitivity with respect to the Li & Ma technique. The method is demonstrated by applying it to Monte Carlo simulations of gamma-ray burst observations with parameters chosen to reproduce the performance of current generation imaging air Cherenkov telescopes (IACTs). One particular example of a simulated burst observation near the current-generation IACT detection threshold results in a sensitivity improvement of approximately 25% . It is also shown that this method can work with highly variable light curves without much computational complexity, and that the sensitivity gain is robust against uncertainties in the *a priori*-defined light curve.

1. Introduction

In high-energy astrophysics, measurements are typically performed in a background-dominated regime [8]. Thus, statistical tools for analysis must be employed to estimate the statistical significance associated with the detection of a source under such conditions. The most widely used of these techniques is the likelihood method of Li and Ma [7], Eq. (17), henceforth referred to as LM. The approach relies on the fact that the event counts collected from source and background measurements are Poisson distributed regardless of their particular time behavior during the observation. Time-varying sources can thus be inferred using this method, but any prior information on the time behavior of these sources, as can be provided by other experimental observations or theoretical predictions, cannot be included. We show that the inclusion of *a priori* temporal information is an important tool in improving the sensitivity for the detection of variable or transient sources.

The method we derive is resistant to systematic uncertainties and does not require detailed modeling of instrument response functions. Note that because this likelihood method is not binned– or rather, the number of bins approaches infinity, as will be described henceforth–it can capture rapidly varying lightcurves without the computational complexity and limited resolution of binned methods. It also provides an elegant test statistic that can be readily compared with the LM test statistic.

Generalizations of the LM method have been introduced in the past (for example, [6]). They generally address the detection of extended sources, and may need to rely on instrument response functions (IRFs) such as the point spread function or the energy reconstruction matrix. To the best of our knowledge, no attempt has been made to derive a test statistic based on *a priori* knowledge of the source light curve. This can be achieved in a straightforward approach since the time-stamping of events is much more accurate (to ∼1 μs) than the variability time scale of any plausible gamma-ray source.

Section 2 begins with a derivation of the LM test statistic Section 2.1, and then gives a derivation of a simplified form of a time-dependent test statistic (Section 2.2). The simplification relies on the assumption that the background rate is time independent. In Section 2.3 we explore methods of dealing with a background rate that is possibly time dependent. The approach is illustrated with a modification of the “ring background” model [2], commonly used in the analysis of IACT observations, which has been designed to adjust to variable background rates.

In Section 3 we demonstrate our method by applying it to Monte Carlo (MC) simulations of gamma-ray burst (GRB) observations with imaging air Cherenkov telescopes (IACTs). In Section 3.1 we simplify the simulations by assuming that the back-

* Corresponding author.
E-mail addresses: omw2107@columbia.edu, weiner.ori@gmail.com (O.M. Weiner).

http://dx.doi.org/10.1016/j.astropartphys.2017.05.004
0927-6505/

ground rate is time-independent. In Section 3.2 we explore the effect of a time-dependent background on the behavior of the test statistic under the null hypothesis.

2. Mathematical derivation

2.1. The Li & Ma likelihood ratio

We will briefly derive the LM test statistic using average background and signal rates as free parameters instead of event counts as used in the original derivation. Our choice will serve as a smoother transition to the time-dependent test statistic.

In the LM method, the following experiment is assumed: an on-source observation is made for time T_{on} and the instrument is later shifted to observe a nearby off-source region, where no signal is expected, for time T_{off}. A test statistic is derived using maximum likelihood estimation,[1] given the ratio of observing times $\alpha = \frac{T_{\text{on}}}{T_{\text{off}}}$ and the number of counts during the observations, N_{on} and N_{off}.

In the maximum likelihood model, the off-source event counts N_{off} are only due to an unknown background rate b whereas the on-source counts N_{on} are explained by an unknown signal rate s in addition to the same background rate.

Defining the time-averaged background and signal rates, $\bar{b}$ and $\bar{s}$, in relation to the expected number of counts: $\bar{b}T_{\text{off}} =< N_{\text{off}} >$, $(\bar{s}+\bar{b})T_{\text{on}} =< N_{\text{on}} >$, the likelihood is given by

$$\begin{aligned}\mathscr{L} &= P(N_{\text{on}}|<N_{\text{on}}>)P(N_{\text{off}}|<N_{\text{off}}>) \\ &= \frac{e^{-(\bar{s}+\bar{b})T_{\text{on}}}\left((\bar{s}+\bar{b})T_{\text{on}}\right)^{N_{\text{on}}}}{N_{\text{on}}!}\frac{e^{-\bar{b}T_{\text{off}}}\left(\bar{b}T_{\text{off}}\right)^{N_{\text{off}}}}{N_{\text{off}}!} \qquad (1)\end{aligned}$$

where $P(N|<N>)$ stands for the probability of observing N counts in a Poisson distribution of expected value $<N>$.

To derive the null hypothesis likelihood, we simply set the signal rate to 0. In this case the average background rate is $\overline{b_0}$ which completely accounts for all counts observed:

$\overline{b_0}T_{\text{off}} =< N_{\text{off}} >$; $\overline{b_0}T_{\text{on}} =< N_{\text{on}} >$.

The null hypothesis likelihood is given by

$$\mathscr{L}_0 = \frac{e^{-\overline{b_0}T_{\text{on}}}\left(\overline{b_0}T_{\text{on}}\right)^{N_{\text{on}}}}{N_{\text{on}}!}\frac{e^{-\overline{b_0}T_{\text{off}}}\left(\overline{b_0}T_{\text{off}}\right)^{N_{\text{off}}}}{N_{\text{off}}!}. \qquad (2)$$

We find the maximum likelihood values for the rates by maximizing the likelihood of both the null and signal models: $\overline{b_0} = \frac{N_{\text{on}}+N_{\text{off}}}{T_{\text{on}}+T_{\text{off}}}$; $\bar{b} = \frac{N_{\text{off}}}{T_{\text{off}}}$; $\bar{s} = \frac{N_{\text{on}}}{T_{\text{on}}} - \frac{N_{\text{off}}}{T_{\text{off}}}$.

The likelihood ratio then simplifies into:

$$\frac{\mathscr{L}_0}{\mathscr{L}} = \left(\frac{\overline{b_0}}{\bar{b}+\bar{s}}\right)^{N_{\text{on}}}\left(\frac{\overline{b_0}}{\bar{b}}\right)^{N_{\text{off}}}. \qquad (3)$$

This expression is equivalent to Eq. (14) in [7]. Wilks' theorem [11] allows us to describe the behaviour of the null likelihood ratio in the regime of high counting statistics. Under the null hypothesis, $\sqrt{-2\log(\mathscr{L}_0/\mathscr{L})}$ is distributed as a Gaussian variable with unit standard deviation.

2.2. Time-dependent signal, time-independent background

To include arrival-time information, as may be advisable for a known time-dependent signal, we divide T_{on} into an arbitrarily large number of bins N of equal length Δt, such that $N\Delta t = T_{on}$. We will require the likelihood model to assign a probability for the number of counts within each bin independently. This will cause the likelihood to approach 0 as $N \to \infty$ because it will factor in the chance that the arrival times fall within specific bins, the number of which approaches infinity. This behaviour will cancel out in the likelihood ratio test, and the resulting test statistic will converge nicely.

In the limit of large N, each time bin will include either a single event, or no events at all. Each of the time bins is independently Poisson distributed with an expectation value approaching 0 as N increases.

Let b denote the time-independent background rate, and $s(t)$ denote the time-dependent signal rate. The background rate b is treated as an unknown to be optimized by maximum likelihood, and there may be similar unknowns within $s(t)$, such as the amplitude or, a "shape parameter", etc. To use Wilks's theorem we must require the signal and null likelihood models to be nested, and thus $s(t)$ must have at least one such unknown, most simply the amplitude. If use of Wilks's theorem is not possible, computer modelling can replace it, and the condition above can be relaxed.

We denote the arrival times of on-source counts as $\{t_{\text{on}}\} = (t_1, t_2 \ldots t_{n_{on}})$. The likelihood is a product of the Poisson probabilities for the count tally over all N time bins:

$$\begin{aligned}\mathscr{L} &= \left(\prod_{t_i=(\Delta t, 2\Delta t, \ldots N\Delta t)} \frac{[\Delta t(b+s(t_i))]^{\{0,1\}}}{\{0,1\}!}e^{-\Delta t(b+s(t_i))}\right) \\ &\quad \times \frac{(bT_{\text{off}})^{N_{\text{off}}}}{N_{\text{off}}!}e^{-bT_{\text{off}}} \qquad (4)\end{aligned}$$

where {0,1} are chosen depending on whether the are 0 or 1 events in the t_i bin.

$$\begin{aligned}\lim_{N\to\infty}\mathscr{L} &= \Delta t^{N_{\text{on}}}\left(\prod_{t_i\in\{t_{on}\}}(b+s(t_i))\right)\frac{(bT_{\text{off}})^{N_{\text{off}}}}{N_{\text{off}}!} \\ &\quad \times e^{-b(T_{\text{on}}+T_{\text{off}})-\int_0^{T_{\text{on}}} dt s(t)}. \qquad (5)\end{aligned}$$

For the null hypothesis, we set $s(t) = 0$, and denote the background rate as b_0, which will obey essentially the same likelihood ratio as the Li & Ma null hypothesis, with only a change of constants:

$$\mathscr{L}_0 = \Delta t^{N_{\text{on}}} b_0^{N_{\text{on}}}\frac{(b_0T_{\text{off}})^{N_{\text{off}}}}{N_{\text{off}}!}e^{-b_0(T_{\text{on}}+T_{\text{off}})}. \qquad (6)$$

Thus $\mathscr{L}_0$ is also maximized by $b_0 = \frac{N_{\text{on}}+N_{\text{off}}}{T_{\text{on}}+T_{\text{off}}}$, giving

$$\mathscr{L}_0 = \Delta t^{N_{\text{on}}} b_0^{N_{\text{on}}}\frac{(b_0T_{\text{off}})^{N_{\text{off}}}}{N_{\text{off}}!}e^{-(N_{\text{on}}+N_{\text{off}})}. \qquad (7)$$

The last equality follows from introducing the exact form of b_0 into the exponential. The likelihood ratio is given by

$$\frac{\mathscr{L}_0}{\mathscr{L}} = \frac{b_0^{N_{\text{on}}+N_{\text{off}}}}{\left(\prod_{t_i\in\{t_{on}\}}(b+s(t_i))\right)b^{N_{\text{off}}}}e^{b(T_{\text{on}}+T_{\text{off}})+\int_0^{T_{\text{on}}} dt\, s(t)-(N_{\text{on}}+N_{\text{off}})}. \qquad (8)$$

This ratio can be further simplified by exploring the connection between $s(t)$ and b. To do so, we must find the maximum of $\mathscr{L}$ or equivalently of $\log\mathscr{L}$.

For the purpose of detecting a transient source such as a GRB, we will leave only one free parameter, the amplitude, in the signal time profile: $s(t) \triangleq \theta f(t)$, where $f(t)$ is a known time profile of the observed burst, typically $1/t$ [1]. This choice reflects the certainty of the flux decaying rapidly (usually as a power-law), and the uncertainty about the amplitude of VHE emission.

We require both partial derivatives of $\log\mathscr{L}$ to vanish at the maximum of the likelihood function:

$$\frac{\partial\log\mathscr{L}}{\partial b} = \frac{N_{\text{off}}}{b} + \sum_{t_i\in\{t_{on}\}}\frac{1}{b+\theta f(t_i)} - (T_{\text{on}}+T_{\text{off}}) = 0 \qquad (9)$$

[1] Maximum likelihood estimation (MLE) is a method often used to estimate the parameters and significance [9] of astrophysical observations, and is used in the LM method. We will rely on MLE in this paper as well.