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# A simple formula for a planet's mean annual insolation by latitude

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ABSTRACT

In this paper, we use a sixth order Legendre series expansion to approximate the mean annual insolation by latitude of a planet with obliquity angle $\beta$, leading to faster computations with little loss in the accuracy of results. We discuss differences between our method and selected computational results for insolation found in the literature.



## 1. Introduction

Incoming solar radiation is an important input in many earth systems models. This physical quantity is needed in areas ranging from low-dimensional energy balance models (e.g. the Budyko energy balance model (Budyko, 1969)) to large global circulation models (GCMs), e.g. NASA's ModelE AR5 (available on NASA's ModelE website), or earth systems models (ESMs). It is common practice to compute insolation by latitude using computer algorithms. For example, NASA's latitudinal insolation calculations for ModelE AR5 rely on three FORTRAN subroutines that 1) calculate Earth's orbital parameters (eccentricity, obliquity, and longitude of perihelion) as a function of year, 2) calculate distance to the sun and declination angle as functions of time of year and orbital parameters, and 3) calculate the time integrated zenith angle as a function of the declination angle and the time interval of the day.

These computer calculations are useful for models with latitude-longitude grids (as is typical in GCM's or ESM's); however, to convert this information to useable data for other modeling scenarios is not always straightforward. For example, in the Budyko–Widiasih energy balance model, one must know the annual average insolation as a function of latitude in order to make use of the model (Widiasih, 2013). Obtaining such a function by fitting a polynomial, a trigonometric function, or a spline to data points given by a computer program obscures the true relationship between insolation and latitude and may introduce errors that, when integrated over time scales of millennia, give meaningless results.

In the following section we give the results of an integration method used in several sources (Dobrovolskis, 2013; McGehee and Lehman, 2012; Ward, 1974) to find the mean annual insolation by latitude for any planet as a function of obliquity and eccentricity and present our results, a sixth-degree approximation to the insolation distribution. In Section 3 we give examples of this approximation to the insolation distributions of Earth, Mars, and Pluto. We conclude with a discussion of the applicability of these approximations and some interesting mathematical conjectures requiring further investigation.

## 2. Mean annual insolation function

It has been shown in several sources that one can calculate (as a function of latitude) the mean annual insolation of a swiftly rotating planet using only first principles (Dobrovolskis, 2013; McGehee and Lehman, 2012; Ward, 1974). Following the notation in McGehee and Lehman (2012), one can express mean annual insolation $\bar{I}$ as a function of eccentricity $e$, obliquity $\beta$, and sine of latitude $y$ by finding the insolation at any point on the Earth's surface, integrating over the course of one orbital period, then integrating over all longitudes (see McGehee and Lehman, 2012, Section 4). Their results are

$$\bar{I}(e, y, \beta) = Q(e)s(y, \beta)$$

where the distribution of insolation across the sine of the latitude is given by

$$s(y, \beta) = \frac{2}{\pi^2}\int_0^{2\pi}\sqrt{1-(\sqrt{1-y^2}\sin\beta\sin\gamma - y\cos\beta)^2}\,d\gamma \quad (1)$$

(where $\gamma$ is longitude) and the magnitude of insolation is given

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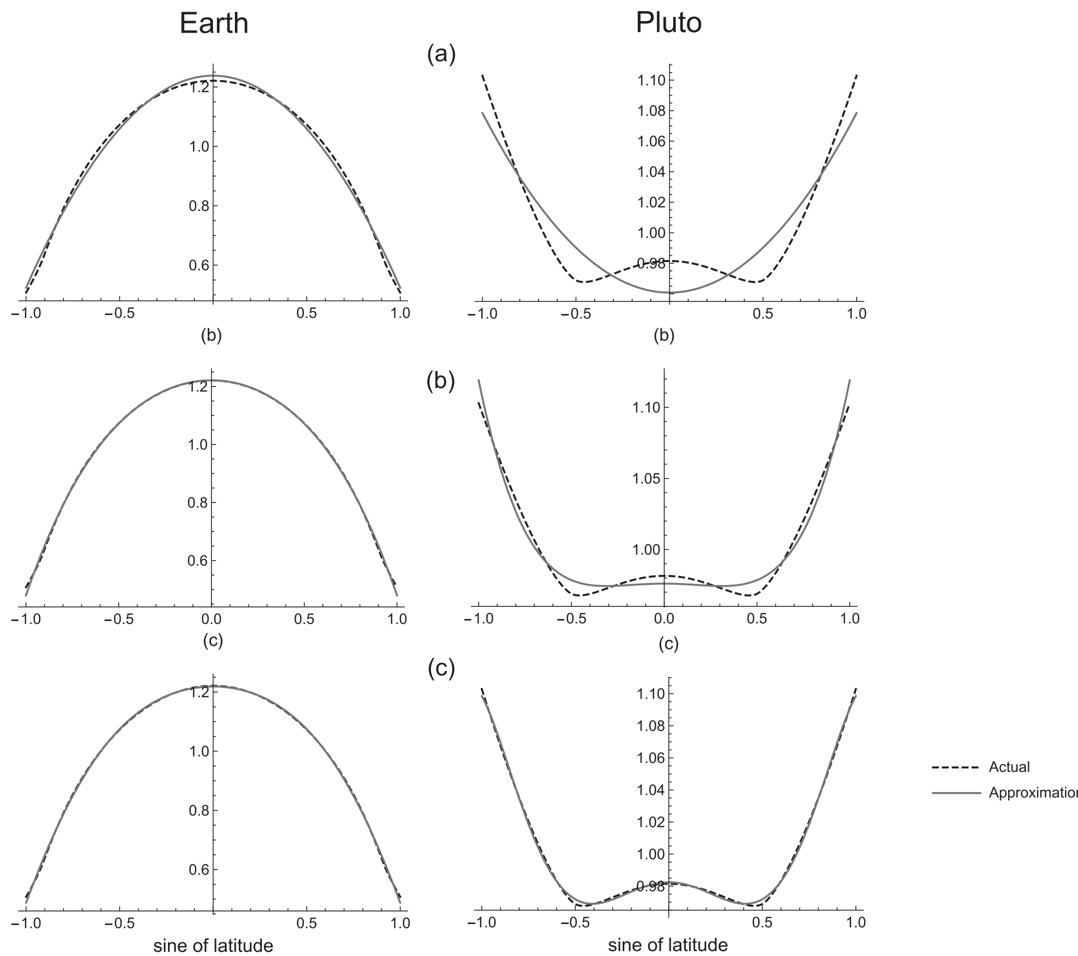


**Fig. 1.** Insolation distributions for Earth and Pluto, both actual (dashed) and various approximations (solid) are shown. In this figure we show (a) second-degree approximations ($\sigma_2$), (b) fourth-degree approximations ($\sigma_4$), and (c) sixth-degree approximations ($\sigma_6$). It is not until the sixth-degree approximation that we capture the slight 'W' shape of Pluto's insolation distribution.

by

$$Q(e) = \frac{Q_0}{\sqrt{1-e^2}}. \tag{2}$$

where $Q_0$ is the global annual average insolation (McGehee and Lehman, 2012). We see that their analysis is general enough to apply to any planet orbiting a star with a spin period much shorter than its orbital period, as long as the appropriate physical parameters are known.

In the appendix we present a recipe for approximating the distribution function $s(y, \beta)$ as a polynomial in $y$ and $\beta$ to any desired degree of accuracy. In Section 3 we show that the sixth-degree approximation is sufficient for most purposes, and is given by

$$\sigma_6(y,\beta) = 1 - \frac{5}{8}p_2(\cos\beta)p_2(y) - \frac{9}{64}p_4(\cos\beta)p_4(y) - \frac{65}{1024}p_6(\cos\beta)p_6(y) \tag{3}$$

where the $p_k$'s are the Legendre polynomials

$$p_2(y) = \left(3y^2-1\right)/2$$
$$p_4(y) = \left(35y^4-30y^2+3\right)/8$$
$$p_6(y) = \left(231y^6-315y^4+105y^2-5\right)/16$$

It should be noted that the polynomial $\sigma_6(y, \beta)$ is the best least-mean-square approximation to the function $s(y, \beta)$. There may be better uniform approximations, but that possibility is not explored here.

North (1975) explicitly gives a second-degree approximation for the insolation distribution for the Earth as

$$\hat{\sigma}_2(y) = 1 - .482p_2(y),$$

stating that the approximation to the actual distribution is accurate to within 2%. North notes that this approximation was first given in Chýlek and Coakley (1975) as a linear interpolation of the insolation distribution, although no closed-form formula is given in that paper. Since this approximation was computed only for the current obliquity of the Earth, it cannot be used to compute changes due to the Milankovitch cycles nor can it be used for other planets. We suggest that the polynomial approximation, $\sigma_6$ given above be used instead of the integral form of the insolation distribution function because the approximation is more computationally efficient and sufficiently accurate to capture the qualitative characteristics of the actual distribution function.

## 3. Planetary examples

The formula for $\sigma_6(y, \beta)$ can be truncated to produce second- and fourth-degree polynomials in $y$ and $\cos\beta$ in the form of

$$\sigma_{2N}(y,\beta) = 1 + \sum_{n=1}^{N} q_{2n}(\beta)p_{2n}(y)$$

for $N = 1, 2, 3$.

As stated in the previous section, North used a second-degree approximation in his analysis of a simple climate model of the