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A variational iteration method for solving nonlinear Lane–Emden problems

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HIGHLIGHTS

- A semi-analytical method for nonlinear Lane–Emden problems.
- Includes adaptive step size and domain decomposition.
- Overcomes the main difficulty arising in the singularity of the equation at the origin point.
- Is simple to implement, accurate when applied to Lane–Emden type equations.

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ABSTRACT

In this paper, an explicit analytical method called the variational iteration method is presented for solving the second-order singular initial value problems of the Lane–Emden type. In addition, the local convergence of the method is discussed. It is often useful to have an approximate analytical solution to describe the Lane–Emden type equations, especially in the case that the closed-form solutions do not exist at all. This convince us that an effective improvement of the method will be useful to obtain a better approximate analytical solution. The improved method is then treated as a local algorithm in a sequence of intervals. Besides, an adaptive version is suggested for finding accurate approximate solutions of the nonlinear Lane–Emden type equations. Some examples are given to demonstrate the efficiency and accuracy of the proposed method.

1. Introduction

Recently, a lot of attention has been focused on the study of singular initial value problems (IVPs) in the second-order ordinary differential equations (ODEs). Many problems in mathematical physics and astrophysics can be modelled by the so-called IVPs of the Lane–Emden type equation (Chandrasekhar, 1967; Davis, 1962; Richardson, 1921):

$$\begin{cases} y'' + \dfrac{2}{x}y' + f(x,y) = g(x), \\ y(0) = a, \quad y'(0) = b, \end{cases} \tag{1}$$

where a and b are constants, $f(x, y)$ is a continuous real valued function, and $g(x) \in C[0, \infty]$. When $f(x,y) = K(y)$, $g(x) = 0$, Eq. (1) reduces to the classical Lane–Emden equation, which was used to model several phenomena in mathematical physics and astrophysics such as the theory of stellar structure, the thermal behavior of a spherical cloud of gas, isothermal gas sphere, and theory of thermionic currents (Chandrasekhar, 1967; Davis, 1962; Richardson, 1921).

The Lane–Emden type equations have significant applications in many fields of scientific and technical world. Therefore various forms of $f(x, y)$ and $g(x)$ have been investigated by many researchers (e.g., Chowdhury and Hashim, 2007; Shawagfeh, 1993; Wazwaz, 2001). A discussion of the formulation of these models and the physical structure of the solutions can be found in the literature. The numerical solution of the Lane–Emden equation (1), as well as other types of linear and nonlinear singular IVPs in quantum mechanics and astrophysics (Krivec and Mandelzweig, 2001), is numerically challenging because of the singularity behavior at the origin $x = 0$. But analytical solutions are more needed to understand physical better. Recently, many analytical methods were used to solve the Lane–Emden equation (He, 2003; Liao, 2003; Yildirim and Ozis, 2007). Those methods are based on either series solutions or perturbation techniques (Bender et al., 1989; Mandelzweig and Tabakin, 2001; Ramos, 2005; 2008). However, the convergence region of the corresponding results is very small.

The variational iteration method (VIM) was first introduced by the Chinese mathematician J.H. He (He, 1999; 1997a; 1997b; 1998; He et al., 1999; He, 2000, 2006; He and Wu, 2007; He, 2007)

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and has been widely applied by many researchers to handle linear and nonlinear problems. The VIM is used in Tatari and Dehghan (2007) to solve some problems in calculus of variations. This technique is used in Ozer (2007) to solve the boundary value problems with jump discontinuities. Authors of Biazar and Ghazvini (2007) applied the variational iteration method to solve the hyperbolic differential equations. This method is employed in Odibat and Momani (2006) to solve the nonlinear differential equations of fractional order. For more and new applications of the method the interested reader is referred to Lu (2015), Lu and Ma (2016), Hu and He (2016).

The strategy being pursued in this work rests mainly on establishing a useful algorithm based on the VIM (He, 1999; Ghorbani and Momani, 2010) to find highly accurate solution of the Lane–Emden type equations, which it

- overcomes the main difficulty arising in the singularity of the equation at $x = 0$.
- is simple to implement, accurate when applied to the Lane–Emden type equations and avoid tedious computational works.

The examples analyzed in the present paper reveal that the newly developed algorithms are easy, effective and accurate to solve the singular IVPs of the Lane–Emden type equation.

2. Description of the method and its convergence

The basic idea of the VIM is constructing a correction functional by a general Lagrange multiplier where the multiplier in the functional could be identified by variational theory (He and Wu, 2007; He, 2007).

Here, the VIM is described for solving Eq. (1). This method provides the solution as a sequence of iterations. It gives convergent successive approximations of the exact solution if such a solution exists, otherwise approximations can be used for numerical purposes.

To explain the basic idea of the VIM, we first consider Eq. (1) as follows:

$$L[y(x)] + N[y(x)] = g(x), \tag{2}$$

with

$$L[y(x)] = y''(x) + \frac{2}{x}y'(x) \quad \text{and} \quad N[y(x)] = f(x, y(x)), \tag{3}$$

where L denotes the linear operator with respect to y and N is a nonlinear operator with respect to y. The basic character of the VIM is to construct a correction functional according to the variational method as:

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(t)\Big(y_n''(t) + \frac{2}{t}y_n'(t) + f(t, \tilde{y}_n(t)) - g(t)\Big)dt, \tag{4}$$

where λ is a general Lagrange multiplier, which can be identified optimally via variational theory, the subscript n denotes the nth approximation, and $\tilde{y}_n$ is considered as a restricted variation, namely $\delta\tilde{y}_n = 0$. Successive approximations, $y_{n+1}(x)$'s, will be obtained by applying the obtained Lagrange multiplier and a properly chosen initial approximation $y_0(x)$. Consequently, the exact solution can be obtained by using

$$y(x) = \lim_{n\longrightarrow\infty} y_n(x). \tag{5}$$

Now, if we want to determine the optimal value of $\lambda(t)$, we continue as follows:

$$\delta y_{n+1}(x) = \delta y_n(x) + \delta\int_0^x \lambda(t)\Big(y_n''(t) + \frac{2}{t}y_n'(t)\Big)dt, \tag{6}$$

which the stationary conditions can be achieved from the relation (6) as:

$$\begin{cases} 1 - \lambda'(x) + \frac{2}{x}\lambda(x) = 0, \\ \lambda(x) = 0, \\ \lambda''(x) - 2\frac{x\lambda'(x) - \lambda(x)}{x^2} = 0, \end{cases} \tag{7}$$

and the Lagrange multiplier is gained via the relation

$$\lambda(t) = -\left(t - \frac{t^2}{x}\right). \tag{8}$$

Finally, the iteration formula can be given as:

$$y_{n+1}(x) = y_n(x) - \int_0^x \left(t - \frac{t^2}{x}\right) \Big(y_n''(t) + \frac{2}{t}y_n'(t) + f(t, y_n(t)) - g(t)\Big)dt. \tag{9}$$

It is interesting to note that for linear Lane–Emden type equations, its exact solution can be obtained easily by only one iteration step due to the fact that the multiplier can be suitably identified, as will be shown in this paper later.

Now we give the following lemma for the iteration formula (9).

Lemma 1. *If $y(x) \in C^2[0, T]$, then, for $x \le T$*

$$\int_0^x \left(t - \frac{t^2}{x}\right)\Big(y''(t) + \frac{2}{t}y'(t)\Big)dt = y(x) - y(0). \tag{10}$$

Proof. The left hand side of the relation (10) can be written as below:

$$\int_0^x \left(t - \frac{t^2}{x}\right)\big(y''(t)\big)dt + \int_0^x \left(2 - \frac{2t}{x}\right)\big(y'(t)\big)dt. \tag{11}$$

Now integrating by parts first integral (11) yields

$$y'(t)\left[t - \frac{t^2}{x}\right]_{t=0}^{t=x} - \int_0^x \left(1 - \frac{2t}{x}\right)\big(y'(t)\big)dt + \int_0^x \left(2 - \frac{2t}{x}\right)\big(y'(t)\big)dt = \int_0^x y'(t)dt = y(x) - y(0), \tag{12}$$

this completes the proof of (10). □

Using (9) and (10), we have the following simple variational iteration formula:

$$y_{n+1}(x) = y(0) - \int_0^x \left(t - \frac{t^2}{x}\right)(f(t, y_n(t)) - g(t))dt. \tag{13}$$

The VIM (13) makes a recurrence sequence $\{y_n(x)\}$ for $x \in [0, T]$. Obviously, the limit of this sequence is the solution of (1) if this sequence is convergent.

Theorem 2. *If $N[y(x)] = f(x, y)$ is Lipschitz-continuous in $[0, T]$ and $g(x) \in C[0, T]$, then the sequence $\{y_n(x)\}$ produced by (13) is convergent for $x \in [0, T]$.*

Proof. In order to prove the sequence $\{y_n(x)\}$ is uniformly convergent to the solution $y(x)$ of (1), we first note that $y_n(x)$ can be written as

$$\begin{aligned} y_n(x) &= y_0(x) + y_1(x) - y_0(x) + \cdots + y_n(x) - y_{n-1}(x) \\ &= y_0(x) + \sum_{j=0}^{n-1}[y_{j+1}(x) - y_j(x)]. \end{aligned} \tag{14}$$

Shortly we show that

$$|y_n(x) - y_{n-1}(x)| \le \frac{N}{L}\frac{(MLx)^n}{n!}, \tag{15}$$