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Biquaternionic reformulation of multifluid plasma equations

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ABSTRACT

Extending the biquaternionic generalization of the Maxwell type equations of compressible fluids, an effort has been introduced for the reformulation of analogous multifluid plasma equations. With the help of an exact correlation between the equations of classical electrodynamics and multifluid plasma model, suitable and elegant biquaternionic expressions including corresponding matrix representations have been presented. The proposed model in this paper generalizes Maxwell type equations of multifluid plasma in a simple, compact and consistent manner. Finally, using the advantages of biquaternions, the wave equation for plasma is reformulated similarly to the compressible fluids.

1. Introduction

In the relevant literature, there are many attempts in order to reformulate fluid equations of motion as a set of well-known equations in electromagnetic theory. These efforts are based on isomorphism between behaviours of fluid dynamics and classical electromagnetism. Logan [1] has made the first attempt to derive a set of Maxwell equations for fluids. Although it was restricted only to the case of the one-dimensional Rayleigh problem, the formally identical equations were obtained for homogeneous conducting media. On the other hand, Troshkin [2] has first introduced analogous equations for incompressible fluids. In the similar paper [3], Marmanis has put forward a new theory of turbulence based on the analogy between electromagnetism and turbulent hydrodynamics in order to describe the dynamical behavior of averaged flow quantities in incompressible fluid flows of high Reynolds numbers.

A general and different formulation from the previous approaches has been presented by Kambe [4,5] because the fluid is regarded as compressible, and the flow field is three dimensional. It has been proved that this system can be reformulated in a form analogous to that of electromagnetism governed by Maxwell's equations with source terms. The electric and magnetic field vectors are defined in terms of the velocity vector and enthalpy which are analogous to the vector potential and scalar potential of electromagnetism. Thus, the vorticity plays the role of magnetic field while the velocity field behaves like the part of a vector potential and the enthalpy (of isentropic flows) plays the part of a scalar potential in electromagnetism. Motivated by experimental measurements of the stress-energy in helical flows or curved stream tubes of turbulent flows, in the consecutive papers [6,7], Scofield and Huq have introduced the fluid dynamical analogs of the electrodynamical Lorentz force law and Poynting theorem. In a recent paper [8], Abreu et al. have considered a charged fluid and its coupling with the electromagnetic field. Obtaining Maxwell type equations for a compressible fluid whose sources are functions of velocity and vorticity, they have analyzed a correlation function and dispersion relation as functions of the Reynolds number.

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Table 1
Multiplication rules of quaternion basis elements.

	$\boldsymbol{e}_0$	$\boldsymbol{e}_1$	$\boldsymbol{e}_2$	$\boldsymbol{e}_3$
$\boldsymbol{e}_0$	1	$\boldsymbol{e}_1$	$\boldsymbol{e}_2$	$\boldsymbol{e}_3$
$\boldsymbol{e}_1$	$\boldsymbol{e}_1$	−1	$\boldsymbol{e}_3$	$-\boldsymbol{e}_2$
$\boldsymbol{e}_2$	$\boldsymbol{e}_2$	$-\boldsymbol{e}_3$	−1	$\boldsymbol{e}_1$
$\boldsymbol{e}_3$	$\boldsymbol{e}_3$	$\boldsymbol{e}_2$	$-\boldsymbol{e}_1$	−1

In spite of the fact that all of the relationships have been considered for incompressible turbulent flow and compressible flow, the application of these analogies has not been widely investigated in relation to plasmas. However in the paper [9], Thompson and Moeller have used detailed fluid equations in order to introduce a set of Maxwell type expressions for a plasma in terms of the species canonical vorticity and its cross product with the species velocity. Because this work can be seen as a new framework for the equations of a multifluid plasma, in their companion paper [10], the previous effort has been extended to form an exact isomorphism between multifluid theory and classical electromagnetism.

There are also several efforts in relevant literature based on hypercomplex numbers to reformulate the basic equations of physics in compact and simpler way. Quaternions, as one of the mentioned hypercomplex numbers, were invented by Hamilton [11] in 1843 in order to extend complex numbers to the three dimensions. Therefore, this special mathematical entity is composed of four components, i.e, one scalar and three imaginary. They also form an associative division algebra. If two quaternions are combined with complex unit i, it is called biquaternions (or complex quaternions). Because this hypercomplex structure has capability to express until eight component mathematical structures, the well-known equations of electromagnetism can be reformulated in terms of biquaternions [12–25]. Moreover, similarly to elegant and compact spacetime algebra formulation of finite range classical electrodynamics in the presence of magnetic monopoles [26–28], biquaternions can also be employed to generalize the Maxwell equations in presence of electric, magnetic sources and massive photons [29–35].

On the other hand, the similarity between Newton's law in gravity and Coulomb's law in electricity leads to introduction of some formulations in linear gravity almost identical to the Maxwell equations in electromagnetic theory. Using the formalism related to gravity-electromagnetism equivalence, the field and wave equations of linear gravity have been reformulated in terms of biquaternions [36–38]. Furthermore, similarly to various attempts in relevant literature based on octonions [39–49], biquaternions have been utilized to unify the theories of linear gravity and electromagnetism in the presence of both electric, magnetic, gravitoelectric and gravitomagnetic charges [50–58].

The formal analogy between fluids and electromagnetism has led scientists to reformulation of fluid equations in terms of hypercomplex numbers as previously given for electromagnetic theory and linear gravity [59–62]. A similar discussion can also be done between the behaviours of plasma and electromagnetism. The purpose of this paper is to derive the compact and elegant expressions for multifluid plasma theory by using the set of Maxwell equations describing both the fluid and electromagnetic behaviour of plasma. Stimulating biquaternionic reformulation of Maxwell-type equations of compressible fluids, the field equations of multifluid plasma have been combined in a single equation. Moreover, the plasma wave equations in terms of potentials have been derived in a form similar to electromagnetic and gravitational counterparts previously derived using biquaternion algebra.

The layout of the paper is as follows: a brief introduction to biquaternion algebra with corresponding matrix representations are given in the next Section 2. In the light of the analogy between behaviour of electromagnetism and fluid mechanics, the Maxwell like equations of a plasma are introduced in Section 3. Multifluid plasma equations are reformulated in terms of biquaternions in Section 4. In this section, the corresponding matrix expressions of the derived biquaternionic plasma equations are also discussed. In the last Section 5, conclusion and perspective of the paper are given.

2. Preliminaries

Quaternions were discovered by Hamilton in 1843 during his attempt to generalize complex numbers to the three dimensions [11]. The basic algebraic form of a quaternion $\mathbf{q}$ is given by

$$\mathbf{q} = q_0\boldsymbol{e}_0 + q_1\boldsymbol{e}_1 + q_2\boldsymbol{e}_2 + q_3\boldsymbol{e}_3 \tag{1}$$

and they can be seen as a hypercomplex number with four real components q_i (i=0,1,2,3). Here $\boldsymbol{e}_0$, $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ are quaternion basis elements that obey the multiplication rules in Table 1.

A real quaternion $\mathbf{q}$ is a linear combination of a scalar q_0 and a spatial vector $\boldsymbol{q} = q_1\boldsymbol{e}_1 + q_2\boldsymbol{e}_2 + q_3\boldsymbol{e}_3$:

$$\mathbf{q} = q_0 + \boldsymbol{q}. \tag{2}$$

Therefore, scalars and spatial vectors are in the subspace of quaternions.

On the other hand, if two quaternions $\mathbf{q} = q_0\boldsymbol{e}_0 + q_1\boldsymbol{e}_1 + q_2\boldsymbol{e}_2 + q_3\boldsymbol{e}_3$ and $\mathbf{q}' = q'_0\boldsymbol{e}_0 + q'_1\boldsymbol{e}_1 + q'_2\boldsymbol{e}_2 + q'_3\boldsymbol{e}_3$ are combined with a complex unit i ($i = \sqrt{-1}$) a biquaternion (complex quaternion) is constructed

$$\mathbf{Q} = \mathbf{q} + i\mathbf{q}' = (q_0 + iq'_0)\boldsymbol{e}_0 + (q_1 + iq'_1)\boldsymbol{e}_1 + (q_2 + iq'_2)\boldsymbol{e}_2 + (q_3 + iq'_3)\boldsymbol{e}_3. \tag{3a}$$