ARTICLE IN PRESS

Journal of Magnetism and Magnetic Materials xx (xxxx) xxxx–xxxx

Contents lists available at ScienceDirect

Journal of Magnetism and Magnetic Materials

journal homepage: www.elsevier.com/locate/jmmm

Modelling of a free-surface ferrofluid flow

M. Habera*, J. Hron

Charles University, Faculty of Mathematics and Physics, Mathematical Institute, Sokolovska 83, Prague, Czech Republic

ARTICLE INFO

Keywords:
Ferrofluid
Level-set
Droplet
Fem
Simulation

ABSTRACT

The Cauchy's stress tensor of a ferrofluid exposed to an external magnetic field is subject to additional magnetic terms. For a linearly magnetizable medium, the terms result in interfacial magnetic force acting on the ferrofluid boundaries. This force changes the characteristics of many free-surface ferrofluid phenomena. The aim of this work is to implement this force into the incompressible Navier-Stokes equations and propose a numerical method to solve them. The interface of ferrofluid is tracked with the use of the characteristic level-set method and additional reinitialization step assures conservation of its volume. Incompressible Navier-Stokes equations are formulated for a divergence-free velocity fields while discrete interfacial forces are treated with continuous surface force model. Velocity-pressure coupling is implemented via the projection method. To predict the magnetic force effect quantitatively, Maxwell's equations for magnetostatics are solved in each time step. Finite element method is utilized for the spatial discretization. At the end of the work, equilibrium droplet shape are compared to known experimental results.

1. Introduction

Free surface fluid flows and processes involved in a fluid behaviour fascinated scientists since the very beginning of the scientific history. Problems as a breakup of a liquid jet, droplet formation and merging, rising bubbles etc. still lack deeper understanding because of a complex and nonlinear equations governing such phenomena. In addition, they play a role in many industrial processes: *fuel injection*, *fibre spinning*, *ink-jet printing*, *etc*.

All these phenomena become even more attractive in terms of *ferrohydrodynamics*. Ferrofluid reacts to a magnetic field and changes its shape and rheology.

This work attempts to present a physical model and numerical method successful in capturing multiphase(free-surface) ferrofluid flow with special focus on the interface development.

2. Theory and methods

2.1. Balance of momentum for ferrofluid

Balance of momentum for continuous medium in Eulerian setting reads

$$\frac{\partial(\rho \mathbf{u})}{\partial t} + \nabla\cdot(\rho \mathbf{u} \otimes \mathbf{u}) = \nabla\cdot\mathbb{T} + \mathbf{f}, \tag{1}$$

with $\mathbf{u}(t, \mathbf{x})\colon I \times \Omega \longrightarrow \mathbb{R}^d$ being Eulerian velocity, a dimension d, $\mathbb{T}$ a Cauchy's stress tensor and $\mathbf{f}(t, \mathbf{x})\colon I \times \Omega \longrightarrow \mathbb{R}^d$ volume density forces.

The continuous medium $\overline{\Omega} := \overline{\Omega_{\text{air}}(t)} \cup \overline{\Omega_{\text{ferr}}(t)}$, $\forall\, t \in I = [0, T]$, $\Omega \in C^{0,1}$, in this work is composed of two immiscible phases - surrounding fluid, e.g. air, $\Omega_{\text{air}}(t)$ and ferrofluid, $\Omega_{\text{ferr}}(t)$.

So called *constitutive relation* must be specified. Here, we adopt several assumptions(simplifications) on the behaviour and properties of the medium, namely

1. the Ω fluid is *incompressible, newtonian*, the stress tensor without magnetic field

   $$\mathbb{T}_{\text{n}} = \widehat{\mathbb{T}}_{\text{n}}(\mathbb{D}) = -m\mathbb{I} + 2\eta\mathbb{D},$$

   with $\mathbb{D} := \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^{\text{T}})$ the rate of deformation tensor, m the Lagrange multiplier ("pressure") enforcing the incompressibility constraint, η dynamic viscosity,
2. the ferrofluid Ω_{ferr} is *linearly magnetizable*, *isotropic* and *homogenous*,
3. there are no macroscopic electric currents and ferrofluid is non-conductive, we work in field of *magnetostatics*,
4. the only effect of magnetic field on ferrofluid is additional *magnetic stress tensor*

   $$\mathbb{T}_{\text{m}} = -\frac{\mu}{2}|\mathbf{H}^2|\mathbb{I} + \mu\mathbf{H} \otimes \mathbf{H}. \tag{2}$$

* Corresponding author.
E-mail addresses: habera@karlin.mff.cuni.cz (M. Habera), hron@karlin.mff.cuni.cz (J. Hron).

http://dx.doi.org/10.1016/j.jmmm.2016.10.045
Received 14 July 2016; Received in revised form 6 October 2016; Accepted 8 October 2016
Available online xxxx
0304-8853/

With the assumptions above the total stress tensor $\mathbb{T} = \widehat{\mathbb{T}}(\mathbb{D}, \mathbf{H}) = \mathbb{T}_n + \mathbb{T}_m$.

The surface tension effect must be included in multiphase flows. We chose so called *continuous surface force*, where surface forces are approximated with volume body forces concentrated on the interface. This allows us to write

$$\mathbf{f} = \mathbf{f}_s + \mathbf{f}_g, \quad \mathbf{f}_s = \sigma\kappa\delta\mathbf{n}, \quad \mathbf{f}_g = -\rho g \mathbf{e}_z,$$

where σ is surface tension coefficient, κ curvature, δ an infinitely smooth representative of the Dirac delta distribution concentrated on the interface, $\mathbf{n}$ outward normal to the interface, density ρ and gravitational acceleration g.

Moreover, according to [8] there exists a tensor $\mathbb{T}_s$, such that

$$\mathbf{f}_s = \nabla\cdot\mathbb{T}_s, \quad \mathbb{T}_s = \sigma\delta(\mathbb{I} - \mathbf{n}\otimes\mathbf{n}).$$

2.2. Magnetostatics

As assumed above, the total stress tensor is function of magnetic field intensity $\mathbf{H}$, it must be found a priori, before the momentum balance is solved. Magnetostatic equations and constitutive relation for *linearly magnetizable* medium are

$$\nabla\cdot\mathbf{B} = 0, \quad \nabla\times\mathbf{H} = \mathbf{0}, \quad \mathbf{B} = \mu\mathbf{H}. \tag{3}$$

For most of the experimentally used ferrofluids the magnetization behaviour is well approximated by linear fit for roughly $|\mathbf{H}| < 50\,\mathrm{kA\cdot m^{-1}}$, according to [10]. This condition is satisfied in all consequent numerical experiments. The dependence $|\mathbf{B}|(|\mathbf{H}|)$ could be assumed in a form of Langevin function, but one must be careful in writing the final weak formulation, because the magnetic pressure p_m is no longer equal $\frac{\mu^2|\mathbf{H}|}{2}$. However, we do not consider such case here.

2.3. Diffused interface, level-set function

So called *characteristic level-set* function is constructed to track the domains $\Omega_{air}(t)$, $\Omega_{ferr}(t)$ and their boundaries.

It is a regularized characteristic function of $\Omega_{ferr}(t)$, such that

$$\phi_\varepsilon \in C^\infty(\overline{\Omega}),\ \phi_\varepsilon = \begin{cases} 0 & \mathbf{x}\in\Omega_{air},\ \mathrm{dist}(\mathbf{x}, \partial\Omega_{air}) > \varepsilon/2, \\ 1 & \mathbf{x}\in\Omega_{ferr},\ \mathrm{dist}(\mathbf{x}, \partial\Omega_{ferr}) > \varepsilon/2, \end{cases}$$

and $\|\phi_\varepsilon - \chi_{\Omega_{ferr}}\|_{L^2(\Omega)} \longrightarrow 0$ as $\varepsilon \longrightarrow 0^+$. The epsilon parameter is the thickness of the diffused interface.

With the aid of this function all *discontinuous piece-wise constant* physical quantities are approximated, therefore

$$\rho = \rho_{ferr}^{\phi_\varepsilon}\rho_{air}^{1-\phi_\varepsilon}, \quad \eta = \eta_{ferr}^{\phi_\varepsilon}\eta_{air}^{1-\phi_\varepsilon}, \quad \mu = \mu_{ferr}^{\phi_\varepsilon}\mu_{air}^{1-\phi_\varepsilon}, \tag{4}$$

for density, dynamic viscosity and absolute permeability respectively. Outer normal to the implicitly represented interface is approximated as $\mathbf{n} \approx -\frac{\nabla\phi_\varepsilon}{\|\nabla\phi_\varepsilon\|}$.

2.4. Advection

Since the evolution of boundaries $\partial\Omega_{ferr}(t)$ and $\partial\Omega_{air}(t)$ is the essence of this method, one must *advect/transport* the level-set function in the velocity field $\mathbf{u}$. It is achieved solving

$$\frac{\partial\phi_\varepsilon}{\partial t} + \mathbf{u}\cdot\nabla\phi_\varepsilon = 0. \tag{5}$$

This distorts the profile of level-set function and deteriorates it's property as regularized characteristic function of the ferrofluid phase. Level-set function is therefore *reinitialized* similarly to [5–7]

$$\frac{\partial\phi_\varepsilon}{\partial\tau} + \nabla\cdot[\phi_\varepsilon(1-\phi_\varepsilon)\mathbf{n}(t_0, \mathbf{x})] = \varepsilon\Delta\phi_\varepsilon, \tag{6}$$

note the change in the term on right-hand-side. We take full laplacian instead of normal derivative.

2.5. Numerical methodology

Generally, evolution equations are here discretized in time with *finite differences* (*FDM*). *Finite element* (*FEM*) method is used to discretize in spatial variable. In the sense of FDM we denote a discrete solution at k-th time level with upper index so $\mathbf{u}^k(\mathbf{x}) \approx \mathbf{u}(t_k, \mathbf{x})$, etc.

Because Cauchy's stress tensor is function of magnetic field, we must seek $\mathbf{H}$ before the momentum balance is solved. Since the Ω domain is simply connected the potential character of magnetic field intensity allows us to define $\mathbf{H} =: -\nabla\xi$.

Weak formulation for the magnetostatics problem (3) is: find $\xi^{k-1} \in V := H_0^1(\Omega)$ such that for all $s \in V$

$$-\int_{\partial\Omega}\mu^{k-1}s\nabla\xi^{k-1}\cdot\mathbf{n}\,\mathrm{d}S + (\mu^{k-1}\nabla\xi^{k-1}, \nabla s) = 0 \tag{7}$$

with $(u, v) := \int_\Omega uv\,\mathrm{d}x$ being the standard $L^2(\Omega)$ scalar product. The term $\nabla\xi^{k-1}\cdot\mathbf{n} = -\mathbf{H}^{k-1}\cdot\mathbf{n}$ is substitued with boundary conditions.

The balance of momentum (1) contains in each time step two unknown quantities, velocity $\mathbf{u}^k$ and "pressure" m^k enforcing the incompressibility constraint $\nabla\cdot u^k = 0$. To overcome this difficulty Chorin's *projection method* is used [5,7,9]. First find tentative $\mathbf{u}_*^k \in \mathbf{V} := H_0^1(\Omega; \mathbb{R}^d)$, such that for all $\mathbf{V} \in \mathbf{V}$

$$\begin{aligned} &\frac{1}{\Delta t}(\rho^{k-1}(\mathbf{u}_*^k - \mathbf{u}^{k-1}), \mathbf{V}) - (\rho^{k-1}\mathbf{u}_*^k \otimes \mathbf{u}_*^k, \nabla\mathbf{V}) \\ &- ((m^{k-1}, \nabla\cdot\mathbf{V}) + \frac{1}{\mathrm{Re}}(\eta^{k-1}(\nabla\mathbf{u}_*^k + (\nabla\mathbf{u}_*^k)^{\mathrm{T}}), \nabla\mathbf{V}) \\ &+ \frac{1}{\mathrm{We}}\left(\|\nabla\phi_\varepsilon^{k-1}\|\left(\mathbb{I} - \frac{\nabla\phi_\varepsilon^{k-1}\otimes\nabla\phi_\varepsilon^{k-1}}{\|\nabla\phi_\varepsilon^{k-1}\|^2}\right), \nabla\mathbf{V}\right) \\ &- \frac{1}{\mathrm{Mg}}\left(\frac{\mu^{k-1}}{2}(|\mathbf{H}|^2)^{k-1}, \nabla\cdot\mathbf{V}\right) + \frac{1}{\mathrm{Mg}}(\mu^{k-1}\mathbf{H}\otimes\mathbf{H}, \nabla\mathbf{V}) = 0 \end{aligned} \tag{8}$$

where Re, We are Reynolds and Weber numbers. The dimensionless number $\mathrm{Mg} := \frac{\rho_{ref}u_{ref}^2}{\mu_{ref}H_{ref}^2}$ is often called Magnetic Reynolds number.

Then, in so called *pressure correction* step find $m^k \in V$ that for all $q \in V$

$$\frac{1}{\Delta t}(\nabla\cdot\mathbf{u}_*^k, q) + \frac{m_{ref}}{x_{ref}u_{ref}\rho_{ref}}\left(\frac{\nabla(m^k - m^{k-1})}{\rho^{k-1}}, \nabla q\right) = 0. \tag{9}$$

Finally, correct the velocity to have zero divergence by finding $\mathbf{u}^k \in \mathbf{V}$ that for all $\mathbf{V} \in \mathbf{V}$

$$\frac{1}{\Delta t}(\mathbf{u}^k - \mathbf{u}_*^k, \mathbf{V}) + \frac{m_{ref}}{x_{ref}u_{ref}\rho_{ref}}\left(\frac{\nabla(m^k - m^{k-1})}{\rho^{k-1}}, \mathbf{V}\right) = 0. \tag{10}$$

We use *implicit Euler* schema for the advection and reinitialization equation. Dimensionless weak formulation for the advection problem (5) is: find $\phi_\varepsilon^k \in V := H_0^1(\Omega)$ such that for all $\varphi \in V$

$$\frac{1}{\Delta t}(\phi_\varepsilon^k - \phi_\varepsilon^{k-1}, \varphi) - \frac{u_{ref}}{x_{ref}}(\phi_\varepsilon^k, \mathbf{u}^k\cdot\nabla\varphi) = 0. \tag{11}$$

Dimensionless weak formulation for the reinitialization (6): find $\phi_\varepsilon^n \in V$ such that for all $\varphi \in V$

$$\frac{1}{\Delta\tau}(\phi_\varepsilon^n - \phi_\varepsilon^{n-1}, \varphi) - \left(\phi_\varepsilon^n(1-\phi_\varepsilon^n), \frac{\nabla\phi_\varepsilon^k}{\|\nabla\phi_\varepsilon^k\|}\cdot\nabla\varphi\right) + \varepsilon(\nabla\phi_\varepsilon^n, \nabla\varphi) = 0. \tag{12}$$

In the Fig. 1 we give a simple overview of the method used.

Weak formulations and mesh generation are implemented into FEniCS library [3].