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Gaussian impurity moving through a Bose-Einstein superfluid

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ABSTRACT

In this paper a finite Gaussian impurity moving through an equilibrium Bose-Einstein condensate at $T = 0$ is studied. The problem can be described by a Gross-Pitaevskii equation, which is solved perturbatively. The analysis is done for systems of 2 and 3 spatial dimensions. The Bogoliubov equation solutions for the condensate perturbed by a finite impurity are calculated in the co-moving frame. From these solutions the total energy of the perturbed system is determined as a function of the width and the amplitude of the moving Gaussian impurity and its velocity. In addition we derive the drag force the finite sized impurity approximately experiences as it moves through the superfluid, which proves the existence of a superfluid phase for finite extensions of the impurities below the speed of sound. Finally we find that the force increases with velocity until an inflection point from which it decreases again in 2 and 3d.

1. Introduction

One fundamental property of Landau's theory of superfluidity is that below a certain critical velocity a impurity moves dissipationless through the superfluid, since excitations are energetically out of reach [1]. In the early 20th century this idea has been found to be realised in Bose-Einstein condensates (BEC) occupying the ground state of a many-body system below a critical temperature [2–5]. Later Helium II was considered to represent such a state of dissipationless flow [6], while more recently superfluidity was confirmed for various weakly interacting and dilute gases of atoms or molecules at ultra-low temperatures forming a BEC in the nano Kelvin range [7–12]. Only a few years ago the direct observation of superfluidity in terms of frictionless flow in 2d Bose gases has been observed experimentally [13]. Besides superfluidity has been observed in exotic quasi-Bose-Einstein condensates such as those of Exciton-Polaritons at temperatures close to the Kelvin regime or even at room temperature [14,15]. Theoretically in the semiclassical mean-field regime the corresponding order parameter of the BEC describing all the atoms in the ground state, the so called condensate wave function, is governed by the Gross-Pitaevskii equation (GPE) [16,17].

Utilising this GPE framework here we study the behaviour of the superfluid by the well-established scenario of an obstacle in relative motion to the Bose-Einstein superfluid [18–20]. While a superfluid phase can be expected for low velocities above a critical velocity of relative motion between the obstacle and the superfluid a drag force arises due to the possibility of emission of elementary excitations that break down the superfluid phase [21]. For the scenario of a small impurity it was shown that the emergence of the doublet is the first excitation to break down the superfluid phase at a critical velocity below the speed of sound, while even smaller obstacles do not change the critical velocity at all and so it equals the speed of sound of the Bose-Einstein condensate [22]. In general the quantitative investigation of critical velocities in BEC can be used to probe the superfluidity and beyond criticality various additional topological phase transitions of the fluid associated with certain velocities [21,23].

In 2 spatial dimensions for example the excitation creation and the critical velocity of the relative motion is comparable to the situation of a rotating BEC, where below a critical rotation velocity the condensate is vortex free, while above various phases from vortex generation to giant vortices have been reported [24–30]. Thus it is a matter of motion of the condensate, the form of the perturbing impurity and dimensionality, at which velocities of the impurity excitations are generated and what kind of excitations [31,32]. Furthermore the physical dimension of the Bose-Einstein condensate sets the stage for the possible excitations a superfluid can carry, from dark solitons in 1d to vortices in 2d to vortex rings in 3d [31].

The mathematical description of an impurity in a quantum gas is non-trivial and various approaches to this topic exist. E.g. in [33] the effect of a pointwise obstacle in a Fermi gas has been studied in detail using various techniques in several interaction regimes, while the works of [34] focuses on the weakly-interacting and [35] on the strongly-interacting regime. Furthermore we note that in particular the drag force depending on the nature of the obstacle and superfluidity has been studied in an oblate 2d condensate in [36] numerically and the analytics of a stirring laser are given in [37] and furthermore in

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[38] a recent experimental investigation has been conducted. In [22] an asymptotic analysis has determined the criticality of the superfluid phase transitions due to a small impurity and in [21] a perturbative approach involving Bogoliubov equations has been studied. In this paper we follow the approach of [21] and analyse the system for a finite weakly interacting Gaussian impurity with a certain amplitude and width. This Gaussian perturbation is moving at a constant velocity through the BEC, while thereby perturbing the condensate. We will investigate the wave generated by the perturbation and the onset and magnitude of the drag force acting upon the impurity as it reaches a critical velocity. Finally we note that our theoretical treatment includes the results presented in [21] as a limiting case, when the Gaussian becomes the Dirac delta function, i.e. a point like impurity without extension.

This paper is structured as follows. First I will determine the linear waves generated by the perturbation moving through the BEC as function of the impurity parameters for general dimensions. Secondly the energy of the perturbed system will be specified in 2d and 3d. Then explicit formulas for the drag force depending on dimensionality will be obtained. Finally I will discuss the results.

2. Linear waves and energy of a small moving Gaussian impurity

2.1. Gross-Pitaevskii equation

In the second quantization scheme the time evolution of the time dependent quantum field operators $\widehat{\Psi}_N^\dagger(\mathbf{r},t)$ describing N interacting Bosons, with corresponding field operators, is governed by the Heisenberg equation [23]

$$i\hbar\partial_t\widehat{\Psi}_N(\mathbf{r},t) = [\widehat{\Psi}_N(\mathbf{r},t), \widehat{H}(\mathbf{r},t)], \tag{1}$$

and the Hamiltonian given by

$$\widehat{H} = \int\left(\frac{\hbar^2}{2m}\nabla\widehat{\Psi}_N^\dagger\nabla\widehat{\Psi}_N\right)d\mathbf{r} + \frac{1}{2}\int\widehat{\Psi}_N^\dagger(\mathbf{r})\widehat{\Psi}_N^\dagger(\mathbf{r}_*)V_{int}(\mathbf{r}_*-\mathbf{r})\widehat{\Psi}_N(\mathbf{r})\widehat{\Psi}_N(\mathbf{r}_*)d\mathbf{r}_* \, d\mathbf{r} + \int\widehat{\Psi}_N^\dagger(\mathbf{r})V(\mathbf{r})\widehat{\Psi}_N(\mathbf{r})d\mathbf{r}. \tag{2}$$

Here m is the mass of the particle. To simplify the interaction potential in (2) one can assume highly local interactions between particles within the condensate, so the interaction potential is approximated by

$$V_{int}(\mathbf{r}-\mathbf{r}_*) = g\delta(\mathbf{r}-\mathbf{r}_*), \tag{3}$$

with interaction strength $g = \frac{4\pi\hbar^2 a}{m}$ and a the scattering length associated with the condensed particle scattering. Using (3), (2) and (1) we get

$$\begin{aligned} i\hbar\partial_t\widehat{\Psi}_N(\mathbf{r},t) &= [\widehat{\Psi}_N(\mathbf{r},t), \widehat{H}(\mathbf{r},t)] = \\ &= \left(-\frac{\hbar^2\nabla^2}{2m} + \int d\mathbf{r}_*\widehat{\Psi}_N(\mathbf{r}_*)V_{int}(\mathbf{r}_*-\mathbf{r})\widehat{\Psi}_N(\mathbf{r}_*) + V(\mathbf{r})\right)\widehat{\Psi}_N = \\ &= \left(-\frac{\hbar^2\nabla^2}{2m} + V(\mathbf{r}) + g|\widehat{\Psi}_N|^2\right)\widehat{\Psi}_N. \end{aligned} \tag{4}$$

By assuming the field operator to be a c-number $\widehat{\Psi}_N = \sqrt{n_0}\psi_0$ and setting $\psi_0 = \frac{1}{\sqrt{n_0}}\psi$ the time evolution simplifies to the so called Gross-Pitaevskii equation [17,23]

$$i\hbar\partial_t\psi(\mathbf{r},t) = \left(-\frac{\hbar^2\nabla^2}{2m} + V(\mathbf{r}) + g|\psi|^2\right)\psi(\mathbf{r},t). \tag{5}$$

Here m is the mass of a single particle and $g|\psi|^2$ represents the self-interactions of the particles with a magnitude g within our condensate.

In words, the quantum many-body problem can be significantly simplified by an effective theory in certain parameter regimes [23,39,40]. For dilute and weakly interacting Bose gases in the ground state, which can be formalized by the parameter limit referred to as Gross-Pitaevskii limit, the Gross-Pitaevskii Eq. (5) provides a convenient and exact tool to understand BEC even in a mathematically rigorous sense. Now one defines the GP energy functional as follows. The energy of a Bose-Einstein condensate made of N particles with mass m with self-interaction strength g - the latter depends on dimensionality [39], and which experiences the external potential V is given by the minimum of the Gross-Pitaevskii energy functional [23,39]

$$\mathcal{E}[\phi] = \int\left(\frac{\hbar^2}{2m}|\nabla\phi|^2 + \frac{g}{2}|\phi|^4 + V|\phi|^2\right)d\mathbf{r} \tag{6}$$

i.e.,

$$\inf_\phi \mathcal{E}[\phi] = E, \tag{7}$$

which is naturally the case by inserting a solution to (5) due to being the Euler-Lagrange equation to this problem. Now, it has been shown that in the *GP-limit*, $N\to\infty$ while $N{\cdot}a$ = fixed, energy and density of the GP theory converge to the corresponding quantities of the *quantum* many-body problem as stated in the following rigorous theorem. If $N\to\infty$ while $N{\cdot}a$ = fixed, then $\lim_{N\to\infty}\frac{E_0}{N} = E$ and $\lim_{N\to\infty}n(\vec{r}) = |\phi|^2$ in the weak L^1-sense, where E_0 is the energy of the *quantum* many-body problem.

While naturally GP theory is considered for 3d, reduction to lower dimensions occurs e.g. for strong confinement along a spatial dimension and we refer to the rigorous GP treatment of this topic in [39]. In 3d $g = 4\pi\hbar^2 a/m$ and by definition $V(\mathbf{r})$ are particle-particle and particle-impurity coupling respectively, with a being the corresponding scattering length [21] and the actual form of $V(\mathbf{r})$ is the spatial distribution of the impurity, while for 2 dimensions those definitions are modified, $g_{2d} = \sqrt{2\pi}\hbar^2 a/a_z$, where $a_z = \sqrt{\hbar/m\omega_z}$ is the oscillator length [21,39] and here the z-axis is effectively frozen. Such a scenario is e.g. given by a strongly confined BEC in z-direction [39]. The minimiser of (6), say $\psi(\mathbf{r},t)$, is the so called condensate wave function, i.e. the mode in which all particles of the dilute weakly interacting gas condense [23,39]. The wavefunction maps from $(\mathbf{r},t)\in\mathbb{R}^{d+1}\to\mathbb{C}$. For the sake of simplicity of notation and clarity of structure we will set $\hbar = 1 = 2m$ in what follows. Now by performing a variation of the energy $\mathcal{E}[\psi]$ with respect to ψ^* under the norm preserving constraint $\|\psi\|_2^2 = 1$, corresponding to the conservation of the particle number, we get the Euler-Lagrange equation for the minimiser, the Gross-Pitaevskii equation, which is given by

$$i\frac{\partial\psi}{\partial t} = -\Delta\psi + V\psi + g|\psi|^2\psi - \mu\psi. \tag{8}$$

Here the chemical potential, i.e. the energy needed to add another particle, is $\partial\mathcal{E}[\psi]/\partial N = \mu$ [39], with N being the number of particles. To model the moving Gaussian impurity we set the external potential

$$V = V_0 e^{-\frac{1}{2\sigma^2}(\mathbf{r}-\mathbf{v}t)^2} \tag{9}$$

with σ corresponding to the Gaussians width, which models a inserted atom or a laser beam [23,32]. Experiments of moving obstacles in BEC have successfully shown the onset of a drag force [11,12], which will be discussed theoretically in a later section.

2.2. Linear waves

For a single moving obstacle in any dimension the Gross-Pitaevskii energy functional is given by

$$\mathcal{E}[\psi] = \int\left(|\nabla\psi|^2 + \frac{g}{2}|\psi|^4 + V_0 e^{-\frac{1}{2\sigma^2}(\mathbf{r}-\mathbf{v}t)^2}|\psi|^2\right)d\mathbf{r}. \tag{10}$$

To derive the energy of linear waves and their form we make an ansatz of the form $\psi = \phi_0 + \delta\psi$, where ϕ_0 represents the unperturbed part solving the GPE without potential and $\delta\psi(\mathbf{r},t)$ a small perturbation as a consequence of the Gaussian impurity. By inserting this ansatz in (8)