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Analytic approaches to relativistic hydrodynamics

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Abstract

I summarize our recent work towards finding and utilizing analytic solutions of relativistic hydrodynamic. In the first part I discuss various exact solutions of the second-order conformal hydrodynamics. In the second part I compute flow harmonics v_n analytically using the anisotropically deformed Gubser flow and discuss its dependence on n, p_T, viscosity, the chemical potential and the charge.

Keywords: Relativistic hydrodynamics, analytic solutions, heavy-ion collisions

1. Introducution

Why analytic hydro? The past decade has witnessed a tremendous success of relativistic hydrodynamics in describing observables of heavy-ion collisions [1]. Nowadays, a number of sophisticated numerical codes for solving the hydrodynamic equation exist. Together with the realistic initial condition and the QCD equation of state, they can fit the bulk of heavy-ion data at RHIC and the LHC quite well. In such circumstances, it is easy to get an impression that there is not much one can do analytically.

Yet, there are multiple reasons to study analytic solutions of the hydrodynamic equation. Firstly, they provide physical intuition into the problem. There are famous solutions such as the Hubble flow for the expansion of the universe and the Bjorken flow for the expansion of fireballs in heavy-ion collisions. These solutions, while different from reality in details, are something one always keeps in mind as the zeroth approximation. Secondly, the hydrodynamic equation is an interesting and fascinating subject in its own right from a mathematical viewpoint. Many analytic solutions of the ideal and viscous hydrodynamic equations have been found over a century. Yet, a complete understanding of the Navier-Stokes equation remains one of the most challenging problems of modern mathematics. Thirdly, there are interesting questions which numerical approaches cannot fully answer. For example, 'How do flow harmonics v_n *functionally* depend on n, or viscosity?' It would be interesting if there is a kind of 'pocket formula' for the n-dependence of v_n. Last but not least, analytic solutions are useful for testing the accuracy of numerical codes, especially for viscous hydrodynamics.

In this presentation, I summarize our recent work towards finding and utilizing analytic solutions of relativistic hydrodynamics [2, 3, 4, 5, 6, 7, 8]. The main goal is to demonstrate that there are actually a lot of things one can do analytically. In the first part, I construct exact solutions of the second-order conformal hydrodynamic equation. In the second part, I compute flow harmonics v_n analytically for the anisotropically

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deformed Gubser flow [9, 10]. Some of the results have direct phenomenological implications and are worth pursuing in more elaborate numerical studies.

2. Second-order hydrodynamics

The hydrodynamic equation is the continuity equation for the energy momentum tensor

$$\nabla_\mu T^{\mu\nu} = 0\,, \qquad T^{\mu\nu} = \varepsilon u^\mu u^\nu + p(g^{\mu\nu} + u^\mu u^\nu) + \pi^{\mu\nu}\,. \tag{1}$$

$\pi^{\mu\nu}$ is the shear stress tensor relevant to viscous hydrodynamics. In the Navier-Stokes (first order) approximation, it is simply $\pi^{\mu\nu} = -2\eta\sigma^{\mu\nu}$ where η is the shear viscosity. In the second order approximation, the precise form of $\pi^{\mu\nu}$ is still under active debate, but it typically contains a lot of terms. If one assumes conformal symmetry, the number of terms is reduced [11]. But its most general form is still very complicated

$$\begin{aligned}\pi^{\mu\nu} \;=\;& -2\eta\sigma^{\mu\nu} + \tau_\pi\left(\Delta^\mu_\alpha\Delta^\nu_\beta D\pi^{\alpha\beta} + \frac{4}{3}\vartheta\pi^{\mu\nu}\right) + \lambda_2\pi^{\langle\mu}{}_\lambda\Omega^{\nu\rangle\lambda} \\ & +\lambda_1\pi^{\langle\mu}{}_\lambda\pi^{\nu\rangle\lambda} + \tau_\sigma\left(\Delta^\mu_\alpha\Delta^\nu_\beta D\sigma^{\alpha\beta} + \frac{1}{3}\sigma^{\mu\nu}\vartheta\right) - \tilde{\eta}_3\sigma^{\langle\mu}{}_\lambda\sigma^{\nu\rangle\lambda} - \tau_{\pi\pi}\sigma^{\langle\mu}{}_\lambda\pi^{\nu\rangle\lambda} + \lambda_3\Omega^{\langle\mu}{}_\lambda\Omega^{\nu\rangle\lambda}\,,\end{aligned} \tag{2}$$

where $\Omega^{\mu\nu}$ is the vorticity tensor and $\vartheta = \nabla_\mu u^\mu$ is the expansion. The Israel-Stewart equation corresponds to keeping only the first line. In the second line one may argue that $\pi^{\mu\nu}$ and $-2\eta\sigma^{\mu\nu}$ can be identified. However, this is valid only in the asymptotic Navier-Stokes regime which is not assumed here.

First, I will be interested in finding exact solutions of (1) together with (2). In general, finding analytic solutions of (1) is very difficult even in the ideal case $\pi^{\mu\nu} = 0$. If $\pi^{\mu\nu}$ is given by (2) with all the transport coefficients assumed to be nonvanishing, it seems impossible to make any analytical progress. However, there is a trick. To explain this let me review the Gubser flow

3. Gubser flow

One usually solves (1) in the Cartesian coordinates. or in the 'Rindler' coordinates

$$ds^2 = -dt^2 + dx_1^2 + dx_2^2 + dx_3^2\,. \tag{3}$$

If there is boost-invariance, it is often convenient to work in the 'Rindler' coordinates

$$ds^2 = -d\tau^2 + dx_\perp^2 + x_\perp^2 d\phi^2 + \tau^2 dy^2\,, \tag{4}$$

where $\tau = \sqrt{t^2 - x_3^2}$ is the proper time, $y = \frac{1}{2}\ln\frac{t+x_3}{t-x_3}$ is the spacetime rapidity and $x_\perp = \sqrt{x_1^2 + x_2^2}$. If there is conformal symmetry, one can combine the above coordinate transformation with the Weyl transform of the metric $g_{\mu\nu}(x) \to \Lambda^2(x)\hat{g}_{\mu\nu}(\hat{x})$ and solve the hydrodynamic equation in the $\hat{x}^\mu$ coordinates. Gubser's idea was to choose $\Lambda^2 = \tau^2$ [9] so that

$$d\hat{s}^2 = \frac{ds^2}{\tau^2} = \frac{-d\tau^2 + dx_\perp^2 + x_\perp^2 d\phi^2}{\tau^2} + dy^2 = -d\rho^2 + \cosh^2\rho(d\Theta^2 + \sin^2\Theta d\phi^2) + dy^2\,. \tag{5}$$

The resulting metric is that of the three-dimensional de Sitter space dS_3 and a flat dimension for y. In the last equality, the dS_3 part is written in the so-called global coordinates. In the latter coordinates, Gubser considered the simplest form of the flow velocity $(\hat{u}^\rho, \hat{u}^\Theta, \hat{u}^\phi, \hat{u}^y) = (1, 0, 0, 0)$. With this ansatz, the ideal hydrodynamic equation $\nabla_\mu \hat{T}^{\mu\nu} = 0$ can be solved very easily. The solution is then transformed back to Minkowski space

$$\varepsilon \propto \frac{1}{\tau^{4/3}}\frac{1}{(L^4 + 2(\tau^2 + x_\perp^2) + (\tau^2 - x_\perp^2)^2)^{4/3}}\,. \tag{6}$$

The parameter L can be interpreted as the transverse size of the colliding nuclei. One recognizes that the factor $1/\tau^{4/3}$ is identical to the Bjorken flow, but the solution also has a nontrivial dependence on $x_\perp$. It is a boost-invariant, radially expanding solution. Remarkably, Gubser also derived an exact solution of the Navier-Stokes equation where $\pi^{\mu\nu} = -2\eta\sigma^{\mu\nu}$.