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Evaluating multiple polylogarithm values at sixth roots of unity up to weight six

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Abstract

We evaluate multiple polylogarithm values at sixth roots of unity up to weight six, i.e. of the form $G(a_1, \ldots, a_w; 1)$ where the indices a_i are equal to zero or a sixth root of unity, with $a_1 \neq 1$. For $w \leq 6$, we construct bases of the linear spaces generated by the real and imaginary parts of $G(a_1, \ldots, a_w; 1)$ and obtain a table for expressing them as linear combinations of the elements of the bases.

1. Introduction

In the study of Feynman integrals multiple polylogarithms (MPL), also called hyperlogarithms [1] or Goncharov polylogarithms [2], play an important role. They are defined as iterated integrals over integration kernels $\mathrm{d}t/(t - a_i)$, for some set of numbers a_i. More precisely,

$$G(a_1, \ldots, a_w; z) = \int_0^z \frac{1}{t - a_1} G(a_2, \ldots, a_w; t)\, \mathrm{d}t \tag{1}$$

with $a_i, z \in \mathbb{C}$ and $G(z) = 1$.

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In the special case where $a_i = 0$ for all i, the corresponding integral is divergent and instead one defines

$$G(0,\ldots,0;z) = \frac{1}{n!}\log^n z\,. \tag{2}$$

If $a_w \neq 0$ and $\rho \neq 0$, then $G(\rho a_1,\ldots,\rho a_w;\rho z) = G(a_1,\ldots,a_w;z)$ so that one can express such MPL in terms of $G(\ldots;1)$. The length w of the index vector is called the weight. A very well known and studied subset of MPL are harmonic polylogarithms [3], i.e. MPL with the letters $\{0,-1,1\}$.

MPL are very important functions in the field of evaluating multiloop Feynman integrals because they appear quite naturally and are very often involved in results – see, e.g., recent papers [4–9].

One often needs to evaluate Feynman integrals at special values of some variable corresponding to a physically interesting special value of the Feynman integrals. For example, in the case of massive form factor integrals, a specific value of such a variable corresponds to the threshold of creating a pair of massive particles. This leads to MPL with a certain constant set of indices. In this paper, we focus on the case where the indices a_i are equal either to zero or a sixth root of unity, i.e. taken from the seven-letters alphabet $\{0, r_1, r_3, -1, r_4, r_2, 1\}$ with

$$r_{1,2} = \frac{1}{2}\left(1 \pm \sqrt{3}\,\mathrm{i}\right) = \lambda^{\pm 1}\,, \quad r_{3,4} = \frac{1}{2}\left(-1 \pm \sqrt{3}\,\mathrm{i}\right) = \lambda^{\pm 2}\,, \quad \lambda = e^{\pi \mathrm{i}/3} = r_1\,. \tag{3}$$

This specific case appears in a number of physically interesting problems, e.g. in 3-loop QCD corrections to the rho-parameter [10], in the $H \to Z\gamma$ decay at two loops [11]. Other related references where sixth roots of unity appear include [12–16].

Our project was motivated by the necessity to significantly simplify results of the evaluation of certain massive form factors at three loops [17] and the corresponding vertex master integrals [18], both at general momentum squared and at two-particle threshold. These results were indeed written in terms of the constants of the form $G(a_1,\ldots,a_w;1)$ up to $w = 6$, with the indices from the above seven-letters alphabet. As we will show below, these constants can be expressed in terms of elements of the corresponding bases, i.e. irreducible constants.

On the other hand, studying relations between special values of MPL is an interesting mathematical problem – see, e.g., [2,10,12,14,19–26]. As we will see below, our results provide interesting interconnections with various mathematical results and conjectures.

In the next section, we specify the goal of this paper, and mention previous related results in the literature. In Section 3, we solve various relations for the real and imaginary parts of $G(a_1,\ldots,a_w;1)$ up to $w = 6$ recursively with respect to w. We construct bases of the linear spaces generated by $\mathrm{Re}\,G(a_1,\ldots,a_w;1)$ and $\mathrm{Im}\,G(a_1,\ldots,a_w;1)$ and explain how the table of results expressing them as linear combinations of the elements of the bases was obtained. In Conclusion, we discuss our results and perspectives. Elements of our bases up to weight 6 are described in Appendix A.

2. Preliminaries

We consider MPL $G(a_1,\ldots,a_w;1)$ up to $w = 6$, with indices a_i equal either to zero or a sixth root of unity, i.e. taken from the seven-letters alphabet $\{0, r_1, r_3, -1, r_4, r_2, 1\}$ of eq. (3). We imply that $a_1 \neq 1$, i.e. we consider convergent MPL. Our goal is to express any MPL from this set as a linear combination of some irreducible elements.