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Curved momentum spaces from quantum groups with cosmological constant

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ABSTRACT

We bring the concept that quantum symmetries describe theories with nontrivial momentum space properties one step further, looking at quantum symmetries of spacetime in presence of a nonvanishing cosmological constant Λ. In particular, the momentum space associated to the κ-deformation of the de Sitter algebra in $(1+1)$ and $(2+1)$ dimensions is explicitly constructed as a dual Poisson–Lie group manifold parametrized by Λ. Such momentum space includes both the momenta associated to spacetime translations and the 'hyperbolic' momenta associated to boost transformations, and has the geometry of (half of) a de Sitter manifold. Known results for the momentum space of the κ-Poincaré algebra are smoothly recovered in the limit $\Lambda \to 0$, where hyperbolic momenta decouple from translational momenta. The approach here presented is general and can be applied to other quantum deformations of kinematical symmetries, including $(3+1)$-dimensional ones.

1. Introduction

Recent developments in quantum gravity research have revived and given new substance to the long-forgotten idea that momentum space should have a nontrivial geometry, an intuition originally due to Max Born [1]. After more than a decade since Deformed Special Relativity (DSR) was first proposed [2,3], it is now understood that a nontrivial geometry of momentum space is a general feature of DSR theories [4–8]. This is intimately related with the presence of the Planck energy as a second relativistic invariant (besides the speed of light), that can play the role of a curvature scale of the momentum manifold [9]. Nontrivial properties of momentum space emerge also in $(2+1)$-dimensional quantum gravity, where explicit computations show that the effective description of quantum gravity coupled to point particles is given by a theory with curved momentum space and noncommutative spacetime coordinates [10–13]. Of more direct interest for the results we are going to present here are models of noncommutative geometry, where the space of momenta that are dual to the noncommutative spacetime coordinates is curved [14–17].

Besides finding increasing theoretical support, Planck-scale modifications of the geometry of momentum space are extremely relevant from a phenomenological point of view. In fact, features due to curvature of momentum space are dual to those that in general relativity are ascribed to curvature of spacetime: in the same way as spacetime curvature induces redshift of energy, curvature of momentum space induces a dual redshift, that is, an energy-dependent correction to the time of flight of free particles [18]. Such effects open up a much needed observational window for Planck-scale physics, since they are testable with astrophysical observations [19].

Despite the recent significant theoretical and phenomenological progress just discussed, an important ingredient which is necessary to connect the properties of momentum space to observations is still missing. In fact, all of the models mentioned above are essentially deformations of special relativity: even though spacetime might be nontrivial (e.g. spacetime coordinates might not commute), still it has vanishing curvature. This is clearly a phenomenological shortcoming, since the most promising observations involve propagation of particles over cosmological distances, for which spacetime curvature cannot be neglected [20]. In the past few years several proposals aimed at extending relativistic models with curved momentum space were put forward in order to include nonvanishing spacetime curvature. The first concrete approach [21] focussed on constructing an extension of the Poincaré algebra that includes both the Planck scale and a (constant) space-

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time curvature scale as relativistic invariants. The resulting algebra can be seen as a DSR version of the de Sitter (hereafter dS) algebra of symmetries, but the associated coalgebra was not investigated. Other proposals focussed on developing a unifying description of the whole phase space of free particles moving on a curved spacetime with deformed local Poincaré symmetries [22–26]. The general understanding coming from these approaches is that when both momentum space and spacetime have nonvanishing curvature they become so intertwined that it is not possible to give a neat geometrical description of the properties of momentum space on its own.

In this work we show that this is not necessarily the case. Indeed, we are able to explicitly construct the curved momentum space generated by quantum-deformed spacetime symmetries in presence of a nonvanishing cosmological constant. We achieve this result by enlarging the momentum space so that it is not only the manifold of momenta associated to translations on spacetime, but it also includes the 'hyperbolic' momenta associated to the boost transformations and the angular momenta associated to rotations. Within this construction we can also show that in the vanishing cosmological constant limit the Lorentz sector is not needed because it decouples from the energy–momentum sector, thus recovering previous results in the literature.

While we would like to argue that our results are general, we use the setting of Hopf algebras to present an explicit derivation. Hopf algebras have proved to be a very useful mathematical framework to model DSR effects. The most studied example is the κ-Poincaré Hopf algebra [27–29], the investigation of which provided inspiration and more precise understanding of several features of DSR models. For example, it can be explicitly shown that the manifold of momenta associated to the κ-Poincaré translation generators is a (portion of a) dS manifold, whose curvature is determined by the quantum deformation scale κ [17,30] and whose metric determines the free particle dispersion relation that is indeed compatible with the κ-Poincaré symmetries, thus showing that the phenomenology associated to the κ-Poincaré algebra fits very naturally within the framework of relative locality [17,31].

Here we present a generalization of all these results by working with the κ-deformation of the dS algebra (see [32–38]). The name is due to the fact that in the limit of vanishing cosmological constant Λ one recovers the κ-Poincaré algebra, while in the limit of vanishing quantum deformation parameter $z = 1/\kappa$ one recovers the algebra of symmetries of the dS spacetime. It is worth noticing that it was exactly using this Hopf algebra that the first pioneering investigations concerning the interplay between spacetime and momentum space curvature were undertaken [39].

The Poisson version of the κ-dS Hopf algebra in $(1+1)$ and in $(2+1)$ dimensions is defined in section 2, where it is shown that the main differences with respect to the corresponding κ-Poincaré structures fully arise in the $(2+1)$ setting: whilst in the vanishing cosmological constant limit the translation generators $\{P_0, P_1, P_2\}$ close a Hopf subalgebra, this is no longer the case for the κ-dS algebra, since the cosmological constant mixes the translation and Lorentz sectors within both the coproduct map and the deformed Casimir function. Thus, for nonvanishing Λ it seems natural to consider an enlarged momentum space including also the dual coordinates to the Lorentz generators. This idea allows us to construct the curved (generalized) momentum manifold in the nonvanishing cosmological constant setting as the full dual Poisson–Lie group manifold, whose explicit construction can be achieved through the Poisson version of the 'quantum duality principle' (see [40–43] and references therein).

The κ-dS dual Poisson–Lie groups are explicitly constructed in section 3. In $(1+1)$ dimensions the dual group coordinates are those associated to both the spacetime translations and boosts, and a certain linear action of the dual group on the origin of momentum space generates (half of) a $(2+1)$-dimensional dS manifold M_{dS_3}, spanned by the orbit of the group passing through the origin. In this case, the fact that boosts have the same role in the momentum space as translation generators can be understood since their coproducts have the same formal structure. In $(2+1)$ dimensions one spatial rotation comes into play and the structure of the κ-dS Hopf algebra is apparently much more involved. Nevertheless, the construction of the full dual Poisson–Lie group G^*_Λ gives the clue for the full geometrical description of the associated momentum space. The dual Lie algebra and its associated Poisson–Lie group are explicitly constructed in section 3.2, and the corresponding linear action on the enlarged momentum space can be defined in such a way that the dual rotation generates the isotropy subgroup of the origin of the momentum space. As a consequence, we find that a $(4+1)$-dimensional space of momenta associated to translations and boosts arises as a dual group orbit passing through the origin, and such a space again has the geometry of (half of) a dS manifold M_{dS_5}. Moreover, in the vanishing cosmological constant limit, the Lorentz sector completely decouples both in the dispersion relation and in the coproduct, thus recovering the well-known κ-Poincaré momentum space. The paper ends with a concluding section in which the applicability of the method here presented to the construction of the κ-AdS momentum space is shown, and the keystones for solving the corresponding $(3+1)$-dimensional problem are presented.

2. The κ-dS Poisson–Hopf algebra

Let us start by reviewing the structural properties of the κ-deformation of the $(1+1)$ and $(2+1)$ dS algebra, which will be presented by considering the cosmological constant $\Lambda > 0$ as an explicit parameter whose $\Lambda \to 0$ limit provides automatically the expressions for the κ-Poincaré algebra. In this way, the specific features of the construction leading to the κ-Poincaré momentum space will become transparent, and the proposed path to its nonvanishing cosmological constant generalization will arise in a natural way.

In the subsection on the $(1+1)$-dimensional case we just briefly present the essential formulas, postponing a more in-depth discussion of the relevant features of the κ-dS algebra to the following subsection focussing on the $(2+1)$-dimensional case.

2.1. The $(1+1)$ κ-dS algebra

The (undeformed) Poisson–Hopf dS algebra in $(1+1)$ dimensions is defined by the brackets

$$\{K, P_0\} = P_1, \qquad \{K, P_1\} = P_0, \qquad \{P_0, P_1\} = -\Lambda\, K, \tag{1}$$

where K is the generator of boost transformations, P_0 and P_1 are the time and space translation generators and the (undeformed) coproduct is given by $\Delta_0(X) = X \otimes 1 + 1 \otimes X$, with $X \in \{K, P_0, P_1\}$. The Poisson version of the $(1+1)$ κ-dS quantum algebra [34] is a Hopf algebra deformation of (1), given by

$$\{K, P_0\} = P_1, \qquad \{K, P_1\} = \frac{\sinh(zP_0)}{z}, \qquad \{P_0, P_1\} = -\Lambda\, K, \tag{2}$$

with deformed coproduct map

$$\begin{aligned} \Delta(P_0) &= P_0 \otimes 1 + 1 \otimes P_0, \\ \Delta(P_1) &= P_1 \otimes e^{\frac{z}{2}P_0} + e^{-\frac{z}{2}P_0} \otimes P_1, \\ \Delta(K) &= K \otimes e^{\frac{z}{2}P_0} + e^{-\frac{z}{2}P_0} \otimes K. \end{aligned} \tag{3}$$