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Theoretical analysis of Casimir and thermal Casimir effect in stationary space–time

Anwei Zhang

Department of Physics and Astronomy, Shanghai Jiao Tong University, Shanghai 200240, China

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ABSTRACT

We investigate Casimir effect as well as thermal Casimir effect for a pair of parallel perfectly plates placed in general stationary space–time background. It is found that the Casimir energy is influenced by the 00-component of metric and the corresponding quantity in dragging frame. We give a scheme to renormalize thermal correction to free energy in curved space–time. It is shown that the thermal corrections to Casimir thermodynamic quantities not only depend on the proper temperature and proper geometrical parameters of the plates, but also on the determinant of space–time metric.

The Casimir effect [1] is one of the most interesting consequences of vacuum fluctuations predicted by quantum field theory. It is originally expressed as the attraction between two neutral, perfectly conducting plates in vacuum. In classical electrodynamics, there should be no force between neutral plates. But the quite remarkable result actually depends on Planck's constant. Therefore, this effect is a purely quantum effect, and results from the restriction of allowed modes in vacuum between the boundaries.

The past few years have seen spectacular developments in Casimir effect, both theoretically and experimentally [2]. Space–time with nontrivial topology, is also new element which has been taken into account [3,4]. Though no material boundaries exist, the identification conditions induced by space–time topology restrict the quantum fields modes. Following this line, lots of investigations had been performed on the plates in non-Euclidean topology space–time [5–7].

Recently, some authors have investigated the Casimir effect under the influence of weak gravitational fields [8–14]. Particularly in [10], the Casimir vacuum energy density between plates in a slightly curved, static space–time background was studied. Then in the weak field approximation, Bezerra et al. [13] investigated the renormalized vacuum energy density in the plates which were placed near the surface of a rotating spherical gravitational source. Relaxing the assumption of weak field approximation, the vacuum energy in the cavity moving in a circular equatorial orbit in the exact Kerr space–time geometry was evaluated [15]. And then the thermal corrections in such a case were calculated [16].

The main purpose of this article is to generalize the results in [10,13,15,16] to a more general case: stationary space–time. We will theoretically analyze the general properties of Casimir energy as well as thermal corrections for cavity in such a general space–time. Besides, we will give another renormalization scheme for Casimir thermal corrections in curved space–time.

We start by defining a local Cartesian coordinate frame (x, y, z) attached to a pair of parallel perfectly conducting plates separated by a distance L, with z axis being perpendicular to the plates and the origin located at the center of the apparatus. In such a local frame, the general stationary space–time background metric can be written as

$$ds^2 = g_{00}(z)dt^2 + g_{11}(z)dx^2 + g_{22}(z)dy^2 + g_{33}(z)dz^2 + 2g_{03}(z)dtdz. \qquad (1)$$

As a preliminary attempt, we only consider the case that the metric is dependent on coordinate z. It can be seen that this metric possesses $\partial_t, \partial_x, \partial_y$ as killing vector, so the massless scalar field confined in the plates has the form[1]

$$\phi_n = N_n \exp(-i\omega_n t + ik_x x + ik_y y) \sin\left(\frac{n\pi}{L}z\right) f(z), \qquad (2)$$

E-mail address: awzhang@sjtu.edu.cn.

[1] Throughout the text, the natural units will be used.

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where N_n is a normalization constant, the sine function stems from standing-wave condition and Dirichlet boundary condition which requires the ϕ_n to be zero at the boundaries of the plates, and $f(z)$ is an unknown function of z which can be solved by using Klein–Gordon equation

$$\frac{1}{\sqrt{-g(z)}}\partial_\mu[\sqrt{-g(z)}g^{\mu\nu}(z)\partial_\nu]\phi_n = 0 \tag{3}$$

with $g(z) = \det(g_{\mu\nu}(z))$. After some calculations, we can have $f(z) = \exp(i\omega_n g^{03}/g^{33})$ and

$$\omega_n = \sqrt{\frac{{g^{33}}^2}{{g^{03}}^2 - g^{00}g^{33}}}\sqrt{\frac{g^{11}}{g^{33}}k_x^2 + \frac{g^{22}}{g^{33}}k_y^2 + \left(\frac{n\pi}{L}\right)^2}. \tag{4}$$

Note that for simplicity, we have taken the approximation $\int_{-L/2}^{L/2} k_z dz \approx k_z L$ in standing-wave condition and $\partial_z(\sqrt{-g(z)} \times g^{3\nu}(z)) \approx 0$ in Klein–Gordon equation. These approximations can be fulfilled when the metric is independent of coordinate z or the distance L between the plates is very small [15] so that we only need expand $g_{\mu\nu}(z)$ to zero order $g_{\mu\nu}(0)$.[2] Actually the separation between plates is on atomic or subatomic scale and when $L = 1\mu m$, the plates is separated relatively large [2]. Thus the above calculations are valid in zero-order approximation for realistic plates in the curved space–time background (1).

The parameter N_n can be obtained from the scalar product [17]

$$(\phi_n, \phi_m) = i\int_\Sigma [(\partial_\mu\phi_n)\phi_m^* - \phi_n(\partial_\mu\phi_m^*)]\sqrt{g_s}n^\mu d\Sigma, \tag{5}$$

where $g_s = -g/g_{00}$ is the determinant of the induced metric on space-like hypersurface Σ, n^μ is a time-like unit vector and in our case it is $(\sqrt{g^{00}}, 0, 0, -g_{03}\sqrt{g^{00}}/g_{33})$. Then from orthogonality condition, we arrive at

$$N_n^2 = \frac{g_{00}\sqrt{-g_{00}g_{11}g_{22}g_{33}}}{-g(2\pi)^2 L\omega_n}. \tag{6}$$

Here we have used the following relations:

$$-g = g_{11}g_{22}(g_{03}^2 - g_{00}g_{33}),\ g^{00} = \frac{-g_{33}}{g_{03}^2 - g_{00}g_{33}},\ g^{11} = \frac{1}{g_{11}},$$
$$g^{22} = \frac{1}{g_{22}},\ g^{33} = \frac{-g_{00}}{g_{03}^2 - g_{00}g_{33}},\ g^{03} = \frac{g_{03}}{g_{03}^2 - g_{00}g_{33}}. \tag{7}$$

Now we proceed to investigate Casimir energy in the cavity which should take the form

$$E_0 = \int_V \langle 0|T_{\mu\nu}|0\rangle U^\mu U^\nu \sqrt{g_s}dV, \tag{8}$$

in which U^μ is the 4-velocity of observer and it is $(1/\sqrt{g_{00}}, 0, 0, 0)$ for static observer located at coordinate origin, $\langle 0|T_{\mu\nu}|0\rangle$ is the expected value of energy–momentum tensor and its 00-component reads

$$\langle 0|T_{00}|0\rangle = \sum_n \iint (\partial_t\phi_n\partial_t\phi_n^* - \frac{1}{2}g_{00}g^{\mu\nu}\partial_\mu\phi_n\partial_\nu\phi_n^*)dk_x dk_y. \tag{9}$$

Performing the integral in z in (8), taking necessary variable substitution and introducing an exponential cutoff function so as to renormalize the divergent energy, finally we can obtain the Casimir energy observed at coordinate origin

$$E_0 = \sqrt{\frac{g_{00}}{\hat{g}_{00}}}E_p, \tag{10}$$

where $\hat{g}_{00} = g_{00} - \frac{g_{03}^2}{g_{33}}$ is the 00-component of metric in dragging frame, $E_p = -\frac{\pi^2 S_p}{1440 L_p^3}$, $S_p = \int\int \sqrt{g_{11}g_{22}}dxdy$ and $L_p = \int_{-L/2}^{L/2}\sqrt{-g_{33} + \frac{g_{03}^2}{g_{00}}}dz$ denote proper surface area and proper length of the cavity, respectively, so E_p is the Casimir energy in flat space–time.

From the expression (10), one can see that for comoving observer in static space–time background, the Casimir energy is just the proper value in Minkowski space–time. This basically agrees with the result obtained for the cavity placed in weak gravitational field [10,11,13]. One can also find that when the observer is in stationary rather than static space–time, the observed value will be different with the proper value by a factor depending on space–time background. It can be checked that the result in [15] is only a particular case of (10). The above conclusions can be understood by taking a Coordinate scale transformation $\sqrt{|g_{\mu\mu}|}dx^\mu = dx'^\mu$, then line element becomes $\eta_{\mu\mu}dx'^\mu dx'^\mu + 2\frac{g_{03}}{\sqrt{-g_{00}g_{33}}}dt'dz'$ with $\eta_{\mu\mu}$ being Minkowski metric. In static case that $g_{03} = 0$, the space–time is actually flat in the rescaled Coordinate, thus we can have $E_0 = E_p$. But in the case of $g_{03} \neq 0$, the space–time is still curved after transformation, so generally $E_0 \neq E_p$. We should note that such a Coordinate transformation does not change the Casimir energy which can be easily verified. The above discussion is only limited to comoving observer case. Actually, the observed Casimir energy is dependent on observer. For an arbitrary stationary observer located at point z, this energy should be

$$E_z = \frac{g_{00}}{g_{00}(z)}\sqrt{\frac{g_{00}}{\hat{g}_{00}}}E_p. \tag{11}$$

The above equation can be rewritten as:

$$g_{00}(z)E_z = g_{00}E_0 = const. \tag{12}$$

Thus the Casimir energy observed at one point is inversely proportional to the 00-component of metric at this point, regardless of the space–time is static or stationary.

In practice, Casimir cavity is immersed in thermal bath. So in the next we will take temperature into account and give thermal corrections to the Casimir thermodynamic quantities. We begin with the thermal correction to free energy [18,16]

$$\Delta_T F = \frac{1}{L}\sum_{n=1}^\infty \iint \frac{dk_x dk_y}{(2\pi)^2}\int_V dxdydz\sqrt{g_s}\, T\ln\left(1 - e^{-\omega_n/T}\right). \tag{13}$$

For the convenience of calculation, here we assume the temperature is independent of coordinate, so it can be extracted from the integral. Now substituting (5) into (13) and taking $\tilde{k}_x = \sqrt{\frac{g^{11}}{g^{33}}}k_x$, $\tilde{k}_y = \sqrt{\frac{g^{22}}{g^{33}}}k_y$, $\beta = \sqrt{\frac{{g^{33}}^2}{{g^{03}}^2 - g^{00}g^{33}}}/T$, then after some algebra, we arrive at

$$\Delta_T F = \frac{-g^{33}\sqrt{g_{00}}S_p}{2\pi\sqrt{g^{11}g^{22}}\beta}\sum_{n=1}^\infty\int_0^\infty kdk\ln\left(1 - e^{-\beta\sqrt{k^2 + (\frac{n\pi}{L})^2}}\right) \tag{14}$$

with $k^2 = \tilde{k}_x^2 + \tilde{k}_y^2$. The logarithm in (14) can be written as power series

$$\Delta_T F = \frac{g^{33}\sqrt{g_{00}}S_p}{2\pi\sqrt{g^{11}g^{22}}\beta^3}\sum_{n,m=1}^\infty \frac{1}{m}\int_{n\pi\beta/L}^\infty hdhe^{-mh}, \tag{15}$$

[2] In this letter $g_{\mu\nu}(0)$, $g^{\mu\nu}(0)$, $g(0)$ are abbreviated to $g_{\mu\nu}$, $g^{\mu\nu}$ and g, respectively.