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Anomalous dimensions and the renormalizability of the four-fermion interaction

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ABSTRACT

We show that when the dynamical dimension of the $\bar{\psi}\psi$ operator is reduced from three to two in a fermion electrodynamics with scaling, a $g(\bar{\psi}\psi)^2 + g(\bar{\psi}i\gamma^5\psi)^2$ four-fermion interaction which is dressed by this electrodynamics becomes renormalizable. In the fermion–antifermion scattering amplitude every term in an expansion to arbitrary order in g is found to diverge as just a single ultraviolet logarithm (i.e. no log squared or higher), and is thus made finite by a single subtraction. While not necessary for renormalizability per se, the reduction in the dimension of $\bar{\psi}\psi$ to two leads to dynamical chiral symmetry breaking in the infrared, with the needed subtraction then automatically being provided by the theory itself through the symmetry breaking mechanism, with there then being no need to introduce the subtraction by hand. Since the vector and axial vector currents are conserved, they do not acquire any anomalous dimension, with the four-fermion $(\bar{\psi}\gamma^\mu\psi)^2$ and $(\bar{\psi}\gamma^\mu\gamma^5\psi)^2$ interactions instead having to be controlled by the standard Higgs mechanism.

1. QED and the four-fermion theory

In this paper we study the ultraviolet structure of a fermion gauge theory (generically quantum electrodynamics (QED)) coupled to a scalar plus pseudoscalar four-fermion (FF) theory, with action $I^m_{\rm QED} + I_{\rm FF}$, where

$$I^m_{\rm QED} = \int d^4x\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\gamma^\mu(i\partial_\mu - eA_\mu)\psi - m\bar{\psi}\psi\right],$$
$$I_{\rm FF} = \int d^4x\left[-\frac{g}{2}[\bar{\psi}\psi]^2 - \frac{g}{2}[\bar{\psi}i\gamma^5\psi]^2\right]. \tag{1}$$

In our study we impose only one requirement, namely that the effect of QED photon exchanges is to dress the fermion bilinear $\bar{\psi}\psi$ so that its dimension $d_\theta(\alpha)$ is dynamically reduced from three to two. With such a dressing we are then able to show that the four-fermion interaction becomes renormalizable to all orders in the four-fermion coupling constant g. To be specific, we note that in the generalized Landau gauge the asymptotic renormalized inverse massive fermion propagator $\tilde{S}^{-1}_m(p)$ obeys [1] the Callan–Symanzik equation

$$\left[m\frac{\partial}{m} + \beta(\alpha)\frac{\partial}{\partial\alpha}\right]\tilde{S}^{-1}_m(p) = -m[1-\gamma_\theta(\alpha)]\tilde{\Gamma}^m_S(p,p,0), \tag{2}$$

where $\tilde{\Gamma}^m_S(p,p,0)$ is the renormalized Green's function associated with the insertion of a zero-momentum composite operator $\theta = \bar{\psi}\psi$ into the inverse massive fermion propagator. Asymptotic scaling is achievable if $\beta(\alpha) = 0$, with one then asymptotically having

$$\tilde{S}^{-1}_m(p) = \not{p} - m\left(\frac{-p^2 - i\epsilon}{m^2}\right)^{\frac{\gamma_\theta(\alpha)}{2}} + i\epsilon,$$
$$\tilde{\Gamma}^m_S(p,p,0) = \left(\frac{-p^2 - i\epsilon}{m^2}\right)^{\frac{\gamma_\theta(\alpha)}{2}}, \tag{3}$$

where $\gamma_\theta(\alpha)$ is the anomalous part of $d_\theta(\alpha)$ as defined as $d_\theta(\alpha) = 3 + \gamma_\theta(\alpha)$. The insertion of the $\tilde{\Gamma}^m_S$ vertex into the four-fermion Green's functions will then soften them sufficiently in the ultraviolet to make them renormalizable if $d_\theta(\alpha)$ is reduced from three to two. The possibility that $d_\theta(\alpha)$ would be equal to two has been much discussed in the literature (see e.g. [2–4] and references therein). Moreover the $d_\theta(\alpha) = 2$ condition has been discussed not just in the Abelian theory that we study here but even in the non-Abelian case as well, where it is considered to be a condition for dynamical symmetry breaking in walking technicolor scenarios (see e.g. [5] and references therein).

QED studies that involve photons that are not dressed at all (quenched photons) can be associated with a $\beta(\alpha)$ that is zero identically for any value of α, with (3) then holding for any value

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of α. If one studies the theory in the ladder (planar graph) approximation with such quenched photons one precisely obtains our required $d_\theta(\alpha)=2$ if the coupling constant α takes the value $\alpha=\pi/3$ [6]. One can go beyond the quenched ladder approximation and include quenched non-planar graphs as well, and to all orders in quenched photon exchanges one again finds [7] asymptotic scaling. Moreover, this all-order result holds for all possible values of α weak or strong, and since $d_\theta(\alpha)$ is a continuous function of α one can in principle choose a large enough α so that $d_\theta(\alpha)$ is reduced to two (or one can make the theory even more convergent if $d_\theta(\alpha)$ is reduced to below two). If one wants to include non-quenched photons ($\beta(\alpha)$ then not zero identically) the scaling result of [7] will continue to hold [8] if, as was subsequently reformulated in [1], $\beta(\alpha)$ has a zero away from the origin. With the beta function actually being zero identically if the photon is quenched, in all these cases the beta function vanishes, and thus in all cases one has asymptotic scaling with anomalous dimensions.

Regardless of whether or not the QED beta function might actually vanish away from the origin, the results we present in this paper can be understood as the statement that when quenched photon exchanges cause $d_\theta(\alpha)$ to be reduced from three to two, the four-fermion theory vertices are softened enough to make the four-fermion theory renormalizable. In fact, suppose one has a set of quenched QED exchange graphs for which the four-fermion interaction is made renormalizable. Then including non-quenched QED exchange graphs will not change this since from this point on the QED and four-fermion sectors are both power-counting renormalizable.[1]

The actual analysis that we present in this paper is divided into two parts, a study of QED plus a four-fermion interaction when the chiral symmetry vacuum is unique, and a study of the same theory when there is dynamical symmetry breaking and the vacuum is degenerate, a symmetry breaking that is actually caused by the very same $d_\theta(\alpha)=2$ condition [9–11]. Since dynamical symmetry breaking is an infrared effect, the ultraviolet behavior of the theory is the same in both the unique and degenerate vacuum cases, and to establish renormalizability per se it suffices to study the unbroken vacuum case alone. However, when we do study the broken vacuum case we not only find renormalizability we find finiteness, with the symmetry breaking procedure itself automatically generating the needed counterterms for us.[2]

2. All-order renormalizability of the four-fermion interaction

Since the ultraviolet behavior of QED is not sensitive to mass, to establish the renormalizability of the four-fermion interaction we can drop the mass term in (1) and study the chirally-symmetric $I^0_{\rm QED}+I_{\rm FF}$, where

$$I^0_{\rm QED}=\int d^4x\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\bar\psi\gamma^\mu(i\partial_\mu-eA_\mu)\psi\right],$$
$$I_{\rm FF}=\int d^4x\left[-\frac{g}{2}[\bar\psi\psi]^2-\frac{g}{2}[\bar\psi i\gamma^5\psi]^2\right]. \qquad (4)$$

The Green's functions that are needed for the scattering generated by $I_{\rm FF}$ are those that involve $\bar\psi\psi$ and $\bar\psi i\gamma^5\psi$. The utility of studying the massless fermion limit is that in this limit what had been only asymptotic scaling now becomes scaling at all momenta, both asymptotic and non-asymptotic. In this limit the exact two-point $\bar\psi\psi$ Green's function is given by (see e.g. [10])

$$\langle\Omega_0|T(\bar\psi(x)\psi(x)\bar\psi(0)\psi(0))|\Omega_0\rangle=\mu^{-2\gamma_\theta(\alpha)}(x^2)^{-d_\theta(\alpha)}, \qquad (5)$$

where μ is an off-shell subtraction point that is needed for a massless theory. Fourier transforming then gives

$$\Pi^0_S(q^2)=-i\int\frac{d^4p}{(2\pi)^4}{\rm Tr}\left[\left(\frac{(-p^2)}{\mu^2}\frac{(-(p+q)^2)}{\mu^2}\right)^{\frac{\gamma_\theta(\alpha)}{4}}\right.$$
$$\left.\times\frac{1}{\not p}\left(\frac{(-p^2)}{\mu^2}\frac{(-(p+q)^2)}{\mu^2}\right)^{\frac{\gamma_\theta(\alpha)}{4}}\frac{1}{\not p+\not q}\right]. \qquad (6)$$

On making a Dyson–Wick contraction of the fields on the left-hand side of (5), we obtain

$$\Pi^0_S(q^2)=-i\int\frac{d^4p}{(2\pi)^4}{\rm Tr}\Big[\tilde S_0(p)\tilde\Gamma^0_S(p,p+q,q)$$
$$\times\tilde S_0(p+q)\tilde\Gamma^0_S(p+q,p,-q)\Big], \qquad (7)$$

where $\tilde S_0(p)=1/\not p$, and $\tilde\Gamma^0_S(p,p+q,q)$ is the Green's function associated with the insertion of the $\bar\psi\psi$ vertex with momentum q_μ into the inverse massless fermion propagator. Comparing (6) and (7), and in parallel to (3), we obtain

$$\tilde\Gamma^0_S(p,p+q,q)=\left[\frac{(-p^2)}{\mu^2}\frac{(-(p+q)^2)}{\mu^2}\right]^{\frac{\gamma_\theta(\alpha)}{4}}$$
$$=\tilde\Gamma^0_S(p+q,p,-q), \qquad (8)$$

a relation that we will need in the following.

On using

$$\Pi^n_{j=1}A_j^{-\lambda_j}=\frac{\Gamma(\sum\lambda_j)}{\Pi_j\Gamma(\lambda_j)}\int\Pi_i d\alpha_i\frac{\delta(1-\sum\alpha_i)\Pi_i\alpha_i^{\lambda_i-1}}{(\sum\alpha_iA_i)^{(\sum\lambda_i)}}, \qquad (9)$$

we can evaluate $\Pi^0_S(q^2)$ as given in (7) analytically. For spacelike q^2, the region of interest for renormalization, we can Wick rotate, and with $\gamma_\theta(\alpha)=-1$ and $q_E^2=-q^2$ obtain

$$\Pi^0_S(q^2)=-4i\int\frac{d^4p}{(2\pi)^4}\frac{(-\mu^2)p.(p+q)}{[p^2(p+q)^2]^{3/2}}$$
$$=-\frac{2\mu^2}{\pi^3}\int_0^{\Lambda^2}du^2u^2\int_0^1d\alpha_1\frac{A^{1/2}(u^2-Aq_E^2)}{(u^2+Aq_E^2)^3}, \qquad (10)$$

where $u_\mu=p_\mu+\alpha_1q_\mu$, and $A=\alpha_1(\alpha_1-1)$. With (10) entailing that $\Pi^0_S(q^2)/\mu^2$ is dimensionless, the leading divergence of $\Pi^0_S(q^2)$ is just a single logarithm, viz.

$$\Pi^0_S(q^2)=-\frac{\mu^2}{4\pi^2}\ln\left(\frac{\Lambda^2}{-q^2}\right). \qquad (11)$$

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[1] Starting with [9–11] and [12] it had been suggested that the condition $d_\theta(\alpha)=2$ could lead to the renormalizability of the four-fermion interaction. In this paper this is explicitly shown to be the case to all orders in the four-fermion interaction.

[2] For the purposes of this paper all that is required is asymptotic scaling as exhibited in (3) and (8) below, where ultraviolet convergence requires only that $\gamma_\theta(\alpha)$ be negative (which it perturbatively is). Once we have such ultraviolet convergence, the condition $\gamma_\theta(\alpha)=-1$ then emerges [9–11] as the unique condition under which the vacuum undergoes dynamical symmetry breaking in the infrared. The results of this paper will thus hold in any theory in which (3) and (8) hold and there exists an $\bar\alpha$ for which $\gamma_\theta(\bar\alpha)=-1$. For theories in which we require $\beta(\alpha)$ (or a non-Abelian analog) to vanish at an isolated zero, it would have to vanish at the same $\bar\alpha$. We had noted in [10] that it is an open question as to whether the conditions $\beta(\bar\alpha)=0$, $\gamma_\theta(\bar\alpha)=-1$ are in fact compatible. If there is compatibility between $\beta(\bar\alpha)=0$, $\gamma_\theta(\bar\alpha)=-1$ in QED, the results of this paper then entail the all order in α, all order in g renormalizability of a four-fermion theory coupled to QED.