Contents lists available at ScienceDirect
Physics Letters B
www.elsevier.com/locate/physletb

Dealing with ghost-free massive gravity without explicit square roots of matrices

Alexey Golovnev, Fedor Smirnov

Faculty of Physics, St. Petersburg State University, Ulyanovskaya ul., d. 1, Saint Petersburg 198504, Russia

A R T I C L E I N F O

Article history:
Received 11 January 2017
Received in revised form 21 April 2017
Accepted 24 April 2017
Available online xxxx
Editor: A. Ringwald

A B S T R A C T

In this paper we entertain a simple idea that the action of ghost free massive gravity (in metric formulation) depends not on the full structure of the square root of a matrix but rather on its invariants given by the elementary symmetric polynomials of the eigenvalues. In particular, we show how one can construct the quadratic action around Minkowski spacetime without ever taking the square root of the perturbed matrix. The method is however absolutely generic. And it also contains the full information on possible non-standard square roots coming from intrinsic non-uniqueness of the procedure. In passing, we mention some hard problems of those apocryphal square roots in the standard approach which might be better tackled with our method. The details of the latter are however deferred to a separate paper.

1. Introduction

The theory of General Relativity enjoys a superb agreement with experimental data all over a wide variety of scales. However, in the realm of cosmology we have a number of uneasy points including the origin of Dark Energy and the nature of Dark Matter. It gave rise to a plenitude of attempts to formulate a viable infrared modification of gravity which would hopefully do better in cosmology than GR. In particular, one of such directions which recently became very popular hinges upon giving a mass to the graviton.

The early days of massive gravity witnessed an almost detective story which starts from the original paper by Fierz and Pauli [1] which presented the linearised ghost-free massive deformation around flat space, and goes through infamous vDVZ discontinuity [2,3] of its massless limit, to the potential resolution via Vainshtein mechanism [4,5], and almost simultaneously to the claim of unavoidable reappearance of the ghost at non-linear level [6], and finally to the ultimate proposal by de Rham, Gabadadze and Tolley [7–11]. The model requires an additional (fiducial) metric which can either be Minkowski $\eta_{\mu\nu}$ as in the first papers on the subject, or can be arbitrary [12,13] and even dynamical with its own Einstein–Hilbert term [14] thereby producing a full-fledged bimetric gravity.

An ugly feature of the model is that the interaction potential is made of $\sqrt{g^{-1}f}$, the square root of the matrix $g^{\mu\alpha}f_{\alpha\nu}$ which, strictly speaking, lacks both guaranteed existence (in the class of real matrices) and uniqueness, see also [15,16]. In this paper we present a method of dealing with massive gravity without explicitly taking the square root of the matrix. In Section 2 we describe the action of massive gravity and its second order expansion around flat space. In Section 3 we introduce the formalism of elementary symmetric polynomials of the eigenvalues, and also explain the problems with non-standard square roots in the usual formulation. In Section 4 we apply our method to quadratic gravity around flat space. Finally, in Section 5 we conclude.

2. Massive gravity

We consider the action of massive gravity in the following form:

$$S = \int d^N x\sqrt{-g}\left(R + m^2 \sum_{n=0}^{N} \beta_n e_n\left(\sqrt{g^{-1}\eta}\right)\right) \tag{1}$$

where the spacetime is N-dimensional with metric $g_{\mu\nu}$, R is its scalar curvature, and $e_n(\mathcal{M})$'s are elementary symmetric polynomials of the eigenvalues λ_i of the matrix $\mathcal{M}^{\mu}_{\nu}$:

$$e_n \equiv \sum_{i_1<i_2<\ldots<i_n} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_n} \tag{2}$$

E-mail addresses: agolovnev@yandex.ru (A. Golovnev), sigmar40k@yandex.ru (F. Smirnov).

http://dx.doi.org/10.1016/j.physletb.2017.04.058
0370-2693/

and $e_0 \equiv 1$ by definition. We see that β_0 gives a pure contribution to the cosmological constant, while $e_N(\sqrt{g^{-1}\eta}) = \frac{1}{\sqrt{-g}}$ adds a mere constant to the action, and therefore it is totally irrelevant unless one wants to have a dynamical metric $f_{\mu\nu}$ instead of η for which it would contribute to its own cosmological constant. Terms with $\beta_1, \ldots, \beta_{N-1}$ make up the potential term for the graviton.

Obviously, these polynomials can be described as coefficients in the characteristic polynomial of $\mathcal{M}$:

$$\det(\mathcal{M} - \lambda\mathbb{I}) = \prod_{n=1}^{N}(\lambda_i - \lambda) = \sum_{n=0}^{N}(-\lambda)^{N-n} \cdot e_n(\mathcal{M}). \tag{3}$$

In particular, e_1 is the ordinary trace

$$e_1(\mathcal{M}) = \sum_i \lambda_i = [\mathcal{M}] \tag{4}$$

where $[\mathcal{M}]$ stands for the trace of $\mathcal{M}$. In other words, we have a shorthand notation which reads $[\mathcal{M}] \equiv \mathcal{M}^\mu_\mu$, $[\mathcal{M}]^2 \equiv (\mathcal{M}^\mu_\mu)^2$, $[\mathcal{M}^2] \equiv \mathcal{M}^\mu_\nu \mathcal{M}^\nu_\mu$, etc. Then we have

$$e_2(\mathcal{M}) = \sum_{i<j} \lambda_i \lambda_j = \frac{1}{2}\left(\left(\sum_i \lambda_i\right)^2 - \sum_i \lambda_i^2\right) = \frac{1}{2}\left([\mathcal{M}]^2 - [\mathcal{M}^2]\right). \tag{5}$$

And one can prove a simple recurrent relation

$$e_n(\mathcal{M}) = \frac{1}{n}\sum_{i=1}^{n}(-1)^{i-1}[\mathcal{M}^i] \cdot e_{n-i}(\mathcal{M}), \tag{6}$$

with $e_n = 0$ for $n > N$.

We will be interested only in the case of $N = 4$, for which we get from (6)

$$e_3(\mathcal{M}) = \frac{1}{6}\left([\mathcal{M}]^3 - 3[\mathcal{M}][\mathcal{M}^2] + 2[\mathcal{M}^3]\right) \tag{7}$$

and also

$$e_4(\mathcal{M}) = \frac{1}{24}\Big([\mathcal{M}]^4 - 6[\mathcal{M}]^2[\mathcal{M}^2] + 3[\mathcal{M}^2]^2 + 8[\mathcal{M}][\mathcal{M}^3] - 6[\mathcal{M}^4]\Big) = \det(\mathcal{M}). \tag{8}$$

The relevant parameters are β_1, β_2, and β_3. The mass parameter m corresponds to the mass scale of the graviton if the largest of β_i's (for $i = 1, 2, 3$) is of order one.

In this paper we would be interested in linearised gravity around Minkowski spacetime, so that we take $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with a small perturbation h to the metric. We will raise and lower the indices of h by η. And then $h^{\mu\nu}$ gives the linear variation of g^{-1} with inversed sign $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + \mathcal{O}(h^2)$, or with a better accuracy we have

$$g^{\mu\alpha}\eta_{\alpha\nu} = \delta^\mu_\nu - h^\mu_\nu + h^{\mu\alpha}h_{\alpha\nu} + \mathcal{O}(h^3). \tag{9}$$

In the standard approach, the square root matrix $\sqrt{g^{-1}\eta}$ would be found explicitly assuming the trivial root of the unity matrix: $\sqrt{\mathbb{I}} = \mathbb{I}$. Then the first terms of the Taylor expansion

$$\sqrt{\mathbb{I} - H} = \mathbb{I} - \frac{1}{2}H - \frac{1}{8}H^2 + \mathcal{O}(H^3)$$

with $H = h - h^2 + \mathcal{O}(h^3)$ give the desired result when substituted into (4), (5), (7), and (8):

$$e_1(\sqrt{g^{-1}\eta}) = 4 - \frac{1}{2}h^\mu_\mu + \frac{3}{8}h_{\mu\nu}h^{\mu\nu} + \mathcal{O}(h^3), \tag{10}$$

$$e_2(\sqrt{g^{-1}\eta}) = 6 - \frac{3}{2}h^\mu_\mu + \frac{1}{8}(h^\mu_\mu)^2 + h_{\mu\nu}h^{\mu\nu} + \mathcal{O}(h^3), \tag{11}$$

$$e_3(\sqrt{g^{-1}\eta}) = 4 - \frac{3}{2}h^\mu_\mu + \frac{1}{4}(h^\mu_\mu)^2 + \frac{7}{8}h_{\mu\nu}h^{\mu\nu} + \mathcal{O}(h^3), \tag{12}$$

$$e_4(\sqrt{g^{-1}\eta}) = 1 - \frac{1}{2}h^\mu_\mu + \frac{1}{8}(h^\mu_\mu)^2 + \frac{1}{4}h_{\mu\nu}h^{\mu\nu} + \mathcal{O}(h^3). \tag{13}$$

Of course, the last expression (13) can also be derived from $e_4(\sqrt{g^{-1}\eta}) = \frac{1}{\sqrt{-g}}$ where

$$\sqrt{-g} = 1 + \frac{1}{2}h^\mu_\mu + \frac{1}{8}(h^\mu_\mu)^2 - \frac{1}{4}h_{\mu\nu}h^{\mu\nu} + \mathcal{O}(h^3). \tag{14}$$

Quadratic approximations to the β_i terms in the action (1) are easily given by multiplying (10)–(12) by (14):

$$\sqrt{-g} \cdot e_1(\sqrt{g^{-1}\eta}) = 4 + \frac{3}{2}h^\mu_\mu + \frac{1}{4}(h^\mu_\mu)^2 - \frac{5}{8}h_{\mu\nu}h^{\mu\nu} + \mathcal{O}(h^3), \tag{15}$$

$$\sqrt{-g} \cdot e_2(\sqrt{g^{-1}\eta}) = 6 + \frac{3}{2}h^\mu_\mu + \frac{1}{8}(h^\mu_\mu)^2 - \frac{1}{2}h_{\mu\nu}h^{\mu\nu} + \mathcal{O}(h^3), \tag{16}$$

$$\sqrt{-g} \cdot e_3(\sqrt{g^{-1}\eta}) = 4 + \frac{1}{2}h^\mu_\mu - \frac{1}{8}h_{\mu\nu}h^{\mu\nu} + \mathcal{O}(h^3), \tag{17}$$

$\sqrt{-g} \cdot e_4(\sqrt{g^{-1}\eta}) = 1$ exactly, and of course $\sqrt{-g} \cdot e_0 = \sqrt{-g}$ given by (14).

In this form, the Fierz–Pauli structure of the potential term is not yet obvious. However, we see that there is a non-vanishing first order contribution to the action around Minkowski:

$$V(h) \equiv m^2 \sum_{n=0}^{N} \sqrt{-g} \cdot \beta_n e_n(\sqrt{g^{-1}\eta}) = V(0) + m^2\left(\frac{1}{2}\beta_0 + \frac{3}{2}\beta_1 + \frac{3}{2}\beta_2 + \frac{1}{2}\beta_3\right)h^\mu_\mu + \mathcal{O}(h^2)$$

In order for the flat space to be a solution, we require it vanishes which gives a condition

$$\beta_0 = -3\beta_1 - 3\beta_2 - \beta_3.$$

Being plugged back into the action, it yields the familiar result:

$$V(h) - V(0) = \frac{m^2}{8}(\beta_1 + 2\beta_2 + \beta_3) \cdot \left(h^{\mu\nu}h_{\mu\nu} - (h^\mu_\mu)^2\right) + \mathcal{O}(h^3).$$

Note that we followed the usual path. However, these calculations can be simplified by employing the well-known symmetry of bimetric theory $g_{\mu\nu} \leftrightarrow f_{\mu\nu}$, $\beta_n \leftrightarrow \beta_{N-n}$. It comes from the fact that $e_n(\mathcal{M}^{-1})$ is a polynomial of $\frac{1}{\lambda_i}$ which can be obtained from $e_{N-n}(\mathcal{M})$ by dividing over $\det\mathcal{M}$. In particular,

$$\sqrt{-g} \cdot e_3(\sqrt{g^{-1}\eta}) = e_1(\sqrt{\eta^{-1}g}) = e_1(\sqrt{\mathbb{I} + h}) = 4 + \frac{1}{2}[h] - \frac{1}{8}[h^2] + \mathcal{O}(h^3)$$

which also explains the mysterious disappearance of the $(h^\mu_\mu)^2$-term from $\sqrt{-g} \cdot e_3(\sqrt{g^{-1}\eta})$.