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Duration of classicality in highly degenerate interacting Bosonic systems

Pierre Sikivie *, Elisa M. Todarello

Department of Physics, University of Florida, Gainesville, FL 32611, USA

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ABSTRACT

We study sets of oscillators that have high quantum occupancy and that interact by exchanging quanta. It is shown by analytical arguments and numerical simulation that such systems obey classical equations of motion only on time scales of order their relaxation time τ and not longer than that. The results are relevant to the cosmology of axions and axion-like particles.

The question under consideration here is: on what time scale do highly degenerate, interacting quantum oscillators obey classical equations of motion? Consider the broad class of systems that have a Hamiltonian of the form

$$H = \sum_j \omega_j a_j^\dagger a_j + \frac{1}{4}\sum_{jkln} \Lambda_{jk}^{ln} a_j^\dagger a_k^\dagger a_l a_n \tag{1}$$

where the a_j and $a_j^\dagger$ are annihilation and creation operators satisfying canonical equal-time commutation relations. $\mathcal{N}_j = a_j^\dagger a_j$ is the number of quanta in oscillator j. For the sake of definiteness, we have restricted ourselves in Eq. (1) to systems in which the total number of quanta $N = \sum_j \mathcal{N}_j$ is conserved. The system states are given by linear combinations

$$|\Psi(t)\rangle = \sum_{\{\mathcal{N}_j\}} c(\{\mathcal{N}_j\}, t)\, |\{\mathcal{N}_j\}\rangle \tag{2}$$

of eigenstates $|\{\mathcal{N}_j\}\rangle$ of the $\mathcal{N}_j$ for arbitrary distributions $\{\mathcal{N}_j\} = (\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3, ...)$ of the quanta over the oscillators. In the Heisenberg picture, where the time-dependence of the state vectors has been removed, the annihilation operators $a_j(t)$ satisfy the equations of motion

$$i\dot{a}_j = [a_j, H] = \omega_j a_j + \frac{1}{2}\sum_{kln} \Lambda_{jk}^{ln} a_k^\dagger a_l a_n \,. \tag{3}$$

The classical description of the system is obtained by replacing the $a_j(t)$ with c-numbers $A_j(t)$. They satisfy

$$i\dot{A}_j = \omega_j A_j + \frac{1}{2}\sum_{kln} \Lambda_{jk}^{ln} A_k^* A_l A_n \,. \tag{4}$$

The quantum description always requires vastly more information than the classical one. To be specific, if the number of oscillators is M and the number of quanta N, the classical state is given by $2M - 1$ real numbers, whereas the quantum state is given by

$$D = \frac{(N+M-1)!}{N!(M-1)!} - 1 \tag{5}$$

complex numbers. For example, if $N = 100$ and $M = 10$, $D = 4.26 \cdot 10^{12}$. D increases extremely fast with increasing N and M. Clearly a huge simplification occurs if the system obeys classical equations of motion. The question is: when is this approximation valid?

The question is particularly relevant to axion cosmology [1–6]. The number of axions inside a co-moving volume of size (1 Mpc)^3 today is $N \simeq 4 \cdot 10^{81}$, assuming all the dark matter is axions and the axion mass is 10^{-5} eV. Before structure formation, their momentum dispersion is at most of order $\delta p \sim \frac{1}{t_1}\frac{a(t_1)}{a(t)}$ where $t_1 \sim 10^{-7}$ s is the age of the universe when the axion mass effectively turns on, and $a(t)$ is the cosmological scale factor. Their quantum degeneracy, i.e. the average occupation number of those states that the axions occupy, is thus at least of order $\mathcal{N} \sim 10^{61}$ [2]. Almost all discussion of the cosmology of axions [1,6,7] or axion-like [8] particles assumes that the axion fluid obeys classical field equations. However, it was shown in Refs. [2,3] that the axion fluid thermalizes on a time scale shorter than the age of the universe after the photon temperature has dropped below approximately 500 eV. When the axion fluid thermalizes, it satisfies all conditions for Bose–Einstein condensation and this should therefore be the expected outcome on theoretical grounds. Furthermore it was shown [9] that Bose–Einstein condensation of cold dark matter axions explains precisely and in all respects the observational

* Corresponding author.
E-mail address: sikivie@phys.ufl.edu (P. Sikivie).

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evidence for caustic rings of dark matter in disk galaxies. The evidence is summarized in Ref. [10]. Bose–Einstein condensation is a quantum effect. The argument that cold dark matter axions form a Bose–Einstein condensate was questioned [6] in part on the belief that the cosmic axion fluid satisfies classical field equations as a result of its extremely high degeneracy. This belief is also implicit in the many other discussions of dark matter axions, or axion-like particles, which describe the axion fluid by classical field equations [8]. So, we want to ask: is it true that highly degenerate Bosonic systems obey classical equations of motion merely because they are highly degenerate? And, if they obey classical field equations of motion for a while but not forever, what is the time scale over which classical equations of motion are obeyed?

When the interactions among the oscillators are turned off, i.e. when the $\Lambda^{ln}_{jk} = 0$, and the degeneracy $\mathcal{N}$ is high, a classical description is in fact correct, and accurate to order $1/\mathcal{N}$. Indeed Eqs. (3) and (4) are linear in that case and admit solutions that have identical time dependence. If the expected values $\langle\mathcal{N}_j\rangle \equiv \langle\Psi(t)|\mathcal{N}_j|\Psi(t)\rangle$ and their classical analogues $N_j = A^*_j(t)A_j(t)$ are equal initially, they remain equal ever after. In spite of its apparent "triviality", the non-interacting case describes a large number of interesting phenomena where the system has a non-trivial evolution either because the initial state is a linear superposition of different eigenmodes (e.g. the beating of a double pendulum) or because the oscillation frequencies of the oscillators are time-dependent (e.g. parametric resonance). Such phenomena are described by classical physics when $\mathcal{N}$ is large. The production of cold axions by vacuum realignment in the early universe is a case in point. Because the effect is due to the time dependence of the axion mass and interactions do not play an important role, a classical physics calculation produces a correct estimate of the axion cosmological energy density from vacuum realignment [1]. Perhaps the successes of classical physics when $\Lambda^{ln}_{jk} = 0$ and $\mathcal{N} \to \infty$ has led to a widely held belief that classical physics also gives a good description when $\Lambda^{ln}_{jk} \neq 0$ and $\mathcal{N} \to \infty$.

When $\Lambda^{ln}_{jk} \neq 0$, the $\langle\mathcal{N}_j\rangle$ are time-dependent because quanta jump between oscillators in pairs: one quantum jumps from oscillator l to oscillator j while another quantum jumps from n to k. The classical $N_j(t)$ are also time-dependent when $\Lambda^{ln}_{jk} \neq 0$. The question here is whether the time dependence is the same. Assuming the initial state is far from equilibrium, there exists a time scale τ over which the distribution of the quanta over the oscillators changes completely, i.e. each $\langle\mathcal{N}_j\rangle$ changes by order 100%. We call τ the relaxation time and $\Gamma = 1/\tau$ the relaxation rate. If the system is stable, it will move toward thermal equilibrium on a time scale of order τ. If the system is unstable, it will also move towards thermal equilibrium on a time scale of order τ provided the time scale of instability is long compared to τ.

There is a simple a priori reason to expect the quantum and classical descriptions to deviate from each other on a time scale of order τ. Indeed, the quantum description has the system move towards a Bose–Einstein distribution whereas the classical description has the system move towards a Boltzmann distribution. This argument is compelling but perhaps not precise enough to give us an estimate of the time scale of classicality. It allows the classical description to be valid, for example, on a time scale of order $\tau\log(\mathcal{N})$. For the systems that we are familiar with in the laboratory, mainly superfluid ^{4}He and dilute ultra-cold atoms, the quantum degeneracy is not much larger than one. So we have no compelling guidance from experiment to tell us about the behavior of systems with huge degeneracy such as the cosmic axion fluid with $\mathcal{N} \sim 10^{61}$.

To gain insight, consider the evolution equations for the occupation numbers. There are two cases to consider depending whether $\Gamma < \delta\omega$, where $\delta\omega$ is the energy dispersion, or $\Gamma > \delta\omega$. In the first case, called the particle kinetic regime, we have

$$\dot{\mathcal{N}}_j = \sum_{kln} |\Lambda^{ln}_{jk}|^2 \pi\,\delta(\omega_j + \omega_k - \omega_l - \omega_n)\cdot$$
$$\cdot\ [(\mathcal{N}_j+1)(\mathcal{N}_k+1)\mathcal{N}_l\mathcal{N}_n - \mathcal{N}_j\mathcal{N}_k(\mathcal{N}_l+1)(\mathcal{N}_n+1)] \tag{6}$$

for the operators $\mathcal{N}_j(t)$ in the Heisenberg picture [3], and

$$\dot{N}_j = \sum_{kln} |\Lambda^{ln}_{jk}|^2 \pi\,\delta(\omega_j + \omega_k - \omega_l - \omega_n)\cdot$$
$$\cdot\ [(N_k + N_j)N_l N_n - N_k N_j(N_l + N_n)] \tag{7}$$

for the c-numbers $N_j(t)$ [11]. In the second case, called the condensed regime, we have instead [3]

$$\dot{\mathcal{N}}_j = \frac{i}{2}\sum_{kln}(\Lambda^{jk}_{ln} a^\dagger_l a^\dagger_n a_k a_j - \Lambda^{ln}_{jk} a^\dagger_j a^\dagger_k a_l a_n) \tag{8}$$

and

$$\dot{N}_j = \frac{i}{2}\sum_{kln}(\Lambda^{jk}_{ln} A^*_l A^*_n A_j A_k - \Lambda^{ln}_{jk} A^*_j A^*_k A_l A_n)\,. \tag{9}$$

For a fluid of interacting particles, such as the cosmic axion fluid, the oscillators in Eq. (1) are labeled by the particle momenta $\vec{p} = \frac{2\pi}{L}(n_1, n_2, n_3)$ where the n_r $(r = 1, 2, 3)$ are integers and L is the linear size of a large cubic volume $V = L^3$ in which the associated quantum field satisfies periodic boundary conditions. The oscillator frequencies are $\omega_{\vec{p}} = \frac{p^2}{2m}$ in the non-relativistic limit. In the case of cosmic axions, the relevant interactions are $\lambda\phi^4$ and gravitational, for which the couplings are respectively

$$\Lambda^{\vec{p}_3,\vec{p}_4}_{\lambda\ \vec{p}_1,\vec{p}_2} = \frac{\lambda}{4m^2 V}\delta_{\vec{p}_1+\vec{p}_2,\vec{p}_3+\vec{p}_4} \tag{10}$$

and

$$\Lambda^{\vec{p}_3,\vec{p}_4}_{g\ \vec{p}_1,\vec{p}_2} = -\frac{4\pi G m^2}{V}\left(\frac{1}{|\vec{p}_1 - \vec{p}_3|^2} + \frac{1}{|\vec{p}_1 - \vec{p}_4|^2}\right)\delta_{\vec{p}_1+\vec{p}_2,\vec{p}_3+\vec{p}_4}\,. \tag{11}$$

In the particle kinetic regime, Eqs. (6) and (7) imply relaxation rates of order

$$\Gamma_{\rm pk} \sim n\sigma\,\delta v\,\mathcal{N} \tag{12}$$

where n is the physical space density, δv is the velocity dispersion, and σ is the appropriate cross-section. For $\lambda\phi^4$ interactions, $\sigma_\lambda = \frac{\lambda^2}{64\pi m^2}$. For gravity, the appropriate cross-section is that for large angle scattering, $\sigma_g \sim \frac{4G^2m^2}{(\delta v)^4}$, since forward scattering does not contribute to relaxation. In the condensed regime, Eqs. (8) and Eqs. (9) imply relaxation rates of order

$$\Gamma_{{\rm cr},\lambda} \sim \frac{n\lambda}{4m^2} \quad \text{and} \quad \Gamma_{{\rm cr},g} \sim \frac{4\pi G n}{(\delta v)^2} \tag{13}$$

respectively. The relaxation rate estimates appear very different in the two regimes. However they are related by $\Gamma_{\rm pk} \sim (\Gamma_{\rm cr})^2/\delta\omega$ so that they agree with one another at the inter-regime boundary where $\Gamma = \delta\omega$. Axion dark matter was found [2,3] to thermalize in the condensed regime by their gravitational self-interactions when the photon temperature is of order 500 eV.

Eqs. (6) and (8) for quantum evolution closely resemble their classical counterparts, Eqs. (7) and (9). However, let us point out two significant differences between Eqs. (6) and (7). Similar differences exist between Eqs. (8) and (9). The first and, as it will