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Non-commutative fields and the short-scale structure of spacetime

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ABSTRACT

There is a growing evidence that due to quantum gravity effects the effective spacetime dimensionality might change in the UV. In this letter we investigate this hypothesis by using quantum fields to derive the UV behaviour of the static, two point sources potential. We mimic quantum gravity effects by using non-commutative fields associated to a Lie group momentum space with a Planck mass curvature scale. We find that the static potential becomes finite in the short-distance limit. This indicates that quantum gravity effects lead to a dimensional reduction in the UV or, alternatively, that point-like sources are effectively smoothed out by the Planck scale features of the non-commutative quantum fields.

Recent years have witnessed a surge of interest in the possibility of probing the structure of spacetime in the regime in which quantum gravity effects should become relevant by studying the scaling behaviour of the *spectral dimension*. This way of characterizing the dimensionality of a given space is based on a fictitious diffusion process determined, via the heat equation, by the Laplacian associated to the specific quantum gravity model under consideration [1]. The picture that has emerged, starting with the framework of causal dynamical triangulations [2] and subsequently in a variety of approaches to quantum gravity (see e.g. [3–12]), is that of a *dimensional reduction* at very short, Planckian, scales.

The use of the spectral dimension to explore the short scale structure of spacetime has two drawbacks: it relies on a artificial diffusion process characterized by an unphysical time parameter and can be defined only on Euclidean spaces thus applying only to Wick-rotated versions of the models at stance. This has prompted alternative proposals in which the change of dimensionality at short scales is described in terms of more intuitive or operationally better defined notions. In [13], for example, the authors adopted certain thermodynamic quantities to characterize the dimensionality of space-time while in [14] the change of dimensionality was studied in terms of the emission rate perceived by an accelerated detector. Along these lines a particularly intriguing observation [15] suggests that in models of deformed kinematics at the Planck scale, based on deformed, non-linear, energy–momentum dispersion relations, the running of dimensionality at small scales can be actually captured by a change of the more familiar Hausdorff dimension[1] of *momentum space* in the UV. As noticed in [15] such UV behaviour can be modelled by a non-trivial *integration measure* in four-momentum space. This feature is shared by models of deformed relativistic kinematics based on a Lie group momentum space where the deformed integration measure is determined by the curved geometry of the Lie group manifold. In the most studied examples in the literature such deformed kinematics is related to quantum group deformations of relativistic symmetries and in a configuration space picture to a non-commutative space-time as we briefly review below.

In this letter it is our aim to further explore the short-distance structure of space-time emerging in these models by using the associated non-commutative quantum fields as a probe. We study the behaviour of the potential energy between two point-like sources subject to the interaction mediated by a real scalar field living on the non-trivial momentum space. A central tool in our analysis will be the generating functional of the free quantum scalar field coupled to the sources.

Quite strikingly we obtain that, unlike for ordinary local quantum fields, this potential energy *does not* diverge in the zero-distance limit. This indicates that Planck scale effects encoded in the non-abelian group manifold structure of momentum space introduce an effective dimensional reduction in the UV. We show

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[1] The Hausdorff dimension of momentum space in [15] is given by the scaling of the volume of a ball of radius R, e.g. in D-dimensional Euclidean space $V \sim R^D$ and the Hausdorff dimension is D.

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that such effect does not depend on the choice of kinetic operator of the theory (as it is the case for the analysis of the spectral dimension of these models) and that the UV finiteness of the potential is associated to an effective smearing of the point sources as “seen” by the non-commutative interaction carrier field.

The basic feature of the model of deformed kinematics we will be considering is that momentum space, rather than being ordinary flat Minkowski space, is described by the non-abelian Lie group AN_3, a subgroup of the five dimensional Lorentz group $SO(4,1)$ (see [22] for details). As a manifold this group is a “half” of four-dimensional de Sitter space whose cosmological constant κ^2 is representative of an energy scale which can be identified with the Planck energy. The AN_3 group can be obtained by exponentiating the $\mathfrak{an}_3$ Lie algebra

$$[X_0, X_a] = \frac{i}{\kappa} X_a\,, \quad [X_a, X_b] = 0\,; \quad a = 1, \ldots, 3 \tag{1}$$

known in the literature as κ-Minkowski non-commutative spacetime. Notice that in order for the generators of the Lie algebra to carry dimension of length, one has to introduce the constant κ with dimension of energy. This sets the UV scale associated to the curvature $1/\kappa^2$ of the momentum group-manifold (in the limit $\kappa \to \infty$ the algebra of coordinates becomes abelian and one recovers the usual flat momentum space).

One of the key features of field theories defined on the AN_3 four-momentum space is that ordinary plane waves are replaced by group elements i.e. exponentials of the non-commuting algebra elements (1). For instance, a choice of *normal ordering* for the plane waves can be associated to a given parametrization of the group element [18]. In particular for “time-to-the-right” ordered plane waves

$$e_k = e^{-i\vec{k}\cdot\vec{X}} e^{ik_0 X_0}\,, \tag{2}$$

the real parameters k_0, $\vec{k}$ are coordinates on the AN_3 group momentum space and are known as “bicrossproduct” [19] or “horospherical” coordinates [20].

There is yet another important coordinate system on the AN_3 momentum space, describing its embedding into the five-dimensional Minkowski space. Such “embedding coordinates” are related to the bicrossproduct coordinates by the following coordinate transformation

$$\begin{aligned} p_0 &= \kappa \sinh\left(\tfrac{k_0}{\kappa}\right) + \frac{1}{2\kappa} e^{k_0/\kappa} \vec{k}^2\,, \\ \vec{p} &= e^{k_0/\kappa} \vec{k}\,, \\ p_4 &= \kappa \cosh\left(\tfrac{k_0}{\kappa}\right) - \frac{1}{2\kappa} e^{k_0/\kappa} \vec{k}^2\,. \end{aligned} \tag{3}$$

One can easily check that the embedding coordinates above satisfy the constraints

$$-p_0^2 + \vec{p}^2 + p_4^2 = \kappa^2\,, \quad p_0 + p_4 > 0\,, \tag{4}$$

which define the manifold of the AN_3 group as submanifold of four-dimensional de Sitter space. Notice that taking the flat limit $\kappa \to +\infty$ one has $p_0 \to k_0$, $\vec{p} \to \vec{k}$ but $p_4 \to +\infty$ and therefore p_4 can be identified as the “auxiliary” momentum in embedding coordinates to be considered as a function of energy p_0 and spatial momenta $\vec{p}$ via (4).

The kinematical and relativistic properties of the AN_3 group valued momenta, action of Lorentz transformations and composition of momenta, are described by a Hopf algebra deformation of the Poincaré algebra known as κ-Poincaré [16,17,19]. For the purposes of the present work it will be sufficient to describe the action of translation generators on plane waves and the definition of the (deformed) Casimir mass invariant. Translation generators, as in the ordinary case, act on plane waves as derivatives but in this case the Leibniz rule for acting on products of plane waves will be non-linear and non-symmetric, a typical feature of symmetry generators belonging to a non-trivial Hopf algebra. On a single plane wave $e_k = e^{-i\vec{k}\cdot\vec{X}} e^{ik_0 X_0}$ the translation generators P_μ act according to

$$P_\mu\, e_k = p_\mu(k_0, \vec{k})\, e_k\,, \tag{5}$$

with eigenvalues $p_\mu(k_0, \vec{k})$ given by the first four entries of (3). This action is associated to a type of *non-commutative* differential calculus (the interested reader can consult [21] and [22] for full details on this choice of calculus). Let us just mention in passing that covariance requires such differential calculus to be five-dimensional with $P_\mu \equiv \hat{\partial}_\mu$ acting on plane waves as in (5) and the $\hat{\partial}_4$, the additional derivative, as $\hat{\partial}_4\, e_k = (1 - p_4(k_0, \vec{k}))\, e_k$. There is a natural d'Alembertian operator associated to this five-dimensional calculus which, in terms of the eigenvalues of the translation operators described above, realizes the invariant

$$C(p) = p_0^2 - \vec{p}^{\,2} \tag{6}$$

which formally corresponds to the ordinary relativistic mass Casimir. Such invariant has an intuitive geometrical meaning in terms of sub-manifolds of de Sitter momentum space spanned by hyper-surfaces of constant auxiliary momentum p_4.

A free quantum scalar field defined on the Lie group AN_3 can be constructed in a rather straightforward way in terms of a path integral [23]. Indeed path integrals for fields defined on (several copies) of a Lie group are well known and have been widely studied in the quantum gravity literature under the name of *group field theories* (see [24] for a discussion oriented towards non-commutative models and relevant references). From this point of view the “deformed” quantum fields we are considering here can be seen as a one-dimensional group field theory with a non-trivial kinetic term represented by the Casimir $C(p)$.

As mentioned above, in order to explore the dimensionality of space-time as probed by a non-commutative field, we will study the interaction between two point sources mediated by the exchange of a massless scalar particle. From a path integral point of view this information is encoded in the partition function of the scalar field coupled to external sources which, as in ordinary QFT, represents the vacuum-to-vacuum transition amplitude, $Z \equiv \langle 0| e^{-iHT} |0\rangle$. Such partition function can be computed by functional integration of the action for the field and sources. Our starting point will be the non-commutative partition function written in terms of the star product $\star$ associated with the choice of the non-commutative plane waves (2), introduced in [22] and discussed in details in [25],

$$\begin{aligned} Z[J] = \frac{1}{Z[0]} \int \mathcal{D}\phi\, \exp\bigg(\frac{i}{2} \int d^4x\, (\partial_\mu \phi(x) \star \partial_\mu \phi(x) + \phi(x) \star J(x) \\ + j(x) \star \phi(x)) \bigg), \end{aligned} \tag{7}$$

where ∂_μ are derivatives determined by the covariant calculus discussed above. It turns out that the expression above can be recast in terms of ordinary derivatives and products in a *non-local* form using the relation[2]

$$\psi \star \phi \equiv \psi \sqrt{1 - \frac{\Box}{\kappa^2}}\, \phi + \text{total derivative}\,, \tag{8}$$

[2] It should be noted that, as discussed in [25], this simplification is possible only in the case of bilinear expressions and does not apply to higher order polynomials in fields.