Physics Letters B ••• (••••) •••–•••

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Physics Letters B

www.elsevier.com/locate/physletb

Quaternion based generalization of Chern–Simons theories in arbitrary dimensions

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ARTICLE INFO

Article history:
Received 30 March 2017
Received in revised form 19 May 2017
Accepted 22 May 2017
Available online xxxx
Editor: N. Lambert

ABSTRACT

A generalization of Chern–Simons gauge theory is formulated in any dimension and arbitrary gauge group where gauge fields and gauge parameters are differential forms of any degree. The quaternion algebra structure of this formulation is shown to be equivalent to a three $\mathbb{Z}_2$-gradings structure, thus clarifying the quaternion role in the previous formulation.

1. Introduction

A formulation of gauge theory in terms of differential forms has the advantage that it automatically generates a general coordinate invariant formulation since an explicit metric dependence does not appear. One interesting approach along this line is the applications of Chern–Simons action to 3-dimensional gravity [1].

In the formulation of the standard gauge theory only 1-form gauge fields and 0-form gauge parameters play a role as differential forms. It is natural to ask if one can formulate gauge theories in terms of all the degrees of differential forms. A positive answer was given by one of authors (N.K.) and Watabiki many years back with a graded Lie algebra setting [2]. In this paper we focus on the generalization of the Chern–Simons action to arbitrary dimensions with arbitrary degrees of differential forms as gauge fields and parameters for Lie algebra setting and clarify the origin of the quaternion structure which was discovered in the original formulation [2]. In the present formulation the introduction of graded Lie algebra is not required.

2. The origin of the quaternion structure for a three grading gauge system

When we consider standard Abelian gauge theory with differential forms we identify gauge field as one-form and gauge parameter as zero-form. In this gauge system $\mathbb{Z}_2$-grading structure of even-form and odd-form is present. If we define Λ_+ as a set of even forms and Λ_- as a set of odd forms, we have

$$\lambda_+ \wedge \lambda'_+ = \lambda'_+ \wedge \lambda_+ \in \Lambda_+, \quad \lambda_+ \wedge \lambda_- = \lambda_- \wedge \lambda_+ \in \Lambda_-, \quad \lambda_- \wedge \lambda'_- = -\lambda'_- \wedge \lambda_- \in \Lambda_+, \tag{1}$$

where $\lambda_+, \lambda'_+ \in \Lambda_+$, $\lambda_-, \lambda'_- \in \Lambda_-$ and $\wedge$ is a wedge product. Fermionic and bosonic fields have similar $\mathbb{Z}_2$-grading structure with an obvious correspondence.

Let us consider two types of fields $\Phi_{(a,b,c)}$ and $\mathcal{F}_{(a,b,c)}$ which have a three $\mathbb{Z}_2$-grading structure (a, b, c) with $a, b, c, = 0$ or 1. For simplicity we assume that these fields have Abelian nature. Then we introduce two types of commuting structure with respect to the three gradings:

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http://dx.doi.org/10.1016/j.physletb.2017.05.067

$$\Phi_{(a,b,c)}\Phi'_{(a',b',c')} = (-1)^{aa'+bb'+cc'}\Phi'_{(a',b',c')}\Phi_{(a,b,c)}, \tag{2}$$

$$\mathcal{F}_{(a,b,c)}\mathcal{F}'_{(a',b',c')} = (-1)^{(a+b+c)(a'+b'+c')}\mathcal{F}'_{(a',b',c')}\mathcal{F}_{(a,b,c)}. \tag{3}$$

In (2) the three gradings are independent whereas in (3) there is one global grading corresponding $a+b+c$. Let us introduce an object $\mathbf{q}(a,b,c)$ satisfying the following commuting structure:

$$\mathbf{q}(a,b,c)\mathbf{q}(a',b',c') = (-1)^{aa'+bb'+cc'+(a+b+c)(a'+b'+c')}\mathbf{q}(a',b',c')\mathbf{q}(a,b,c). \tag{4}$$

Then we have the following commuting relation:

$$\begin{aligned}&(\Phi_{(a,b,c)}\mathbf{q}(a,b,c))(\Phi'_{(a',b',c')}\mathbf{q}(a',b',c'))\\ &\quad = (-1)^{(a+b+c)(a'+b'+c')}\big(\Phi'_{(a',b',c')}\mathbf{q}(a',b',c')\big)\big(\Phi_{(a,b,c)}\mathbf{q}(a,b,c)\big),\end{aligned} \tag{5}$$

where we assume $\Phi_{(a,b,c)}$ and $\mathbf{q}(a,b,c)$ are not interacting and thus commuting:

$$\Phi_{(a,b,c)}\mathbf{q}(a,b,c) = \mathbf{q}(a,b,c)\Phi_{(a,b,c)}. \tag{6}$$

We recognize now that a field of the $\mathcal{F}$-type, namely with a single global grading, can be written in terms of a field $\Phi_{(a,b,c)}$ with three distinct gradings as

$$\mathcal{F}_{(a,b,c)} = \Phi_{(a,b,c)}\mathbf{q}(a,b,c). \tag{7}$$

In order to define a product of the $\mathcal{F}$-fields, we have to define a product in the $\mathbf{q}(a,b,c)$ space that satisfies (4). This is given by

$$\mathbf{q}(a,b,c)\mathbf{q}(a',b',c') = (-1)^{aa'+bb'+cc'+a'b+b'c+c'a}\mathbf{q}(a+a',b+b',c+c'), \tag{8}$$

where the sums are defined modulo 2. This product is associative. There are eight possible $\mathbf{q}(a,b,c)$'s for $a,b,c=0,1$. However with respect to their commuting structure (4) and with respect to the product (8) there are only four independent $\mathbf{q}(a,b,c)$. In fact the sign factors in (4) and (8) are invariant for $(a,b,c)\to(a+1,b+1,c+1)$ and the same for (a',b',c'). The units $\mathbf{q}(a,b,c)$ can then be identified in pairs and renamed as:

$$\begin{aligned}&\mathbf{q}(1,1,1)=\mathbf{q}(0,0,0)\equiv\mathbf{1}, \quad \mathbf{q}(1,0,0)=\mathbf{q}(0,1,1)\equiv\mathbf{i},\\ &\mathbf{q}(0,1,0)=\mathbf{q}(1,0,1)\equiv\mathbf{j}, \quad \mathbf{q}(0,0,1)=\mathbf{q}(1,1,0)\equiv\mathbf{k}.\end{aligned} \tag{9}$$

It is easy to recognize now that $\mathbf{1},\mathbf{i},\mathbf{j},\mathbf{k}$ satisfy the quaternion algebra

$$\begin{aligned}&\mathbf{i}^2=\mathbf{j}^2=\mathbf{k}^2=-\mathbf{1},\\ &\mathbf{ij}=-\mathbf{ji}=\mathbf{k}, \ \ \mathbf{jk}=-\mathbf{kj}=\mathbf{i}, \ \ \mathbf{ki}=-\mathbf{ik}=\mathbf{j}.\end{aligned} \tag{10}$$

It is important to notice that eq. (8) defines the quaternion algebra in an unconventional way, different from its standard mathematical introduction, and that it links in an unexpected way the quaternion algebra to the existence of three independent gradings.

We can introduce now two types of $\mathcal{F}$-fields, corresponding respectively to $a+b+c$ odd and even:

$$\begin{aligned}\mathcal{A} &= \mathbf{q}(1,1,1)\Phi_{(1,1,1)}+\mathbf{q}(1,0,0)\Phi_{(1,0,0)}+\mathbf{q}(0,1,0)\Phi_{(0,1,0)}+\mathbf{q}(0,0,1)\Phi_{(0,0,1)}\\ &= \mathbf{1}\Phi_{(1,1,1)}+\mathbf{i}\Phi_{(1,0,0)}+\mathbf{j}\Phi_{(0,1,0)}+\mathbf{k}\Phi_{(0,0,1)},\end{aligned} \tag{11}$$

$$\begin{aligned}\mathcal{V} &= \mathbf{q}(0,0,0)\Phi_{(0,0,0)}+\mathbf{q}(0,1,1)\Phi_{(0,1,1)}+\mathbf{q}(1,0,1)\Phi_{(1,0,1)}+\mathbf{q}(1,1,0)\Phi_{(1,1,0)}\\ &= \mathbf{1}\Phi_{(0,0,0)}+\mathbf{i}\Phi_{(0,1,1)}+\mathbf{j}\Phi_{(1,0,1)}+\mathbf{k}\Phi_{(1,1,0)}.\end{aligned} \tag{12}$$

Assuming that the fields $\Phi_{(a,b,c)}$ are Abelian $\mathcal{A}$ and $\mathcal{V}$ are odd and even elements in a $\mathbb{Z}_2$ commuting algebra:

$$\mathcal{A}\mathcal{A}'=-\mathcal{A}'\mathcal{A}, \quad \mathcal{A}\mathcal{V}=\mathcal{V}\mathcal{A}, \quad \mathcal{V}\mathcal{V}'=\mathcal{V}'\mathcal{V}, \tag{13}$$

where $\mathcal{A}'$ and $\mathcal{V}'$ are defined by $\Phi'(a',b',c')$.

It also follows immediately from (11), (12) and the multiplication rules of quaternions that if we denote by Λ_- and Λ_+ the sets of fields respectively of the $\mathcal{A}$-type and $\mathcal{V}$-type then

$$\mathcal{A}\mathcal{A}'=-\mathcal{A}'\mathcal{A}\in\Lambda_+, \quad \mathcal{A}\mathcal{V}=\mathcal{V}\mathcal{A}\in\Lambda_-, \quad \mathcal{V}\mathcal{V}'=\mathcal{V}'\mathcal{V}\in\Lambda_+. \tag{14}$$

In conclusion $\mathcal{A}$ and $\mathcal{V}$ are anticommuting and commuting quaternionic fields whose component fields $\Phi_{(a,b,c)}$ possess three independent $\mathbb{Z}_2$-gradings whose physical meaning and interpretation may be different. We shall use them in what follows to formulate gauge theories with higher differential forms.

The importance of the different sign choice in (2) and (3) was noticed in [3] for two- and three-gradings formulations of supersymmetric gauge theories.

Please cite this article in press as: A. D'Adda et al., Quaternion based generalization of Chern–Simons theories in arbitrary dimensions, Phys. Lett. B (2017), http://dx.doi.org/10.1016/j.physletb.2017.05.067