Physics Letters B ••• (••••) •••–•••

Contents lists available at ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

# On the complete perturbative solution of one-matrix models

A. Mironov [a,b,c], A. Morozov [b,c]

[a] Lebedev Physics Institute, Moscow 119991, Russia
[b] ITEP, Moscow 117218, Russia
[c] Institute for Information Transmission Problems, Moscow 127994, Russia

ARTICLE INFO

*Article history:*
Received 8 May 2017
Received in revised form 28 May 2017
Accepted 31 May 2017
Available online xxxx
Editor: M. Cvetič

ABSTRACT

We summarize the recent results about complete solvability of Hermitian and rectangular complex matrix models. Partition functions have very simple character expansions with coefficients made from dimensions of representation of the linear group $GL(N)$, and arbitrary correlators in the Gaussian phase are given by finite sums over Young diagrams of a given size, which involve also the well known characters of symmetric group. The previously known integrability and Virasoro constraints are simple corollaries, but no vice versa: complete solvability is a peculiar property of the matrix model (hypergeometric) $\tau$-functions, which is actually a combination of these two complementary requirements.


## 1. Main formulas

Recent studies of the two advanced subjects, Hurwitz partition functions in [1] and rainbow tensor models in [2], led to a spectacular new discovery in the old field of the one-matrix models, which goes far beyond their well-established theory presented in [3] (for alternative descriptions see also [4]). They concern the Hermitian and rectangular complex matrix models given by the extended partition functions, respectively

$$Z_N\{t\} = \frac{\int\limits_{N\times N} dM \exp\left(-\frac{\mu}{2}\operatorname{Tr} M^2 + \sum_k t_k \operatorname{Tr} M^k\right)}{\int_{N\times N} dM \exp\left(-\frac{\mu}{2}\operatorname{Tr} M^2\right)} \tag{1}$$

where the integral runs over the $N\times N$ Hermitian matrices, and

$$Z_{N_1,N_2}\{t\} = \frac{\int\limits_{N_1\times N_2} d^2M \exp\left(-\mu \operatorname{Tr} MM^\dagger + \sum_k t_k \operatorname{Tr}(MM^\dagger)^k\right)}{\int_{N_1\times N_2} d^2M \exp\left(-\mu \operatorname{Tr} MM^\dagger\right)} \tag{2}$$

where the integral is over *rectangular* $N_1\times N_2$ complex matrices. With this choice of action, the second model has a symmetry $U(N_1)\times U(N_2)$, much bigger than just a single $U(N)$ in the case of Hermitian model.

In fact, what have been known so far about these models is that

- $Z_N\{t\}$ and $Z_{N,N}\{t\}$ are actually the $\tau$-functions of the Toda chain, i.e. satisfy the infinite system of bilinear Hirota equations [5]
- $Z_N\{t\}$ and $Z_{N_1,N_2}\{t\}$ satisfy the infinite system of recursive Virasoro constraints (see [6] and [7,8] respectively)

$$\left(-\mu\frac{\partial}{\partial t_{n+2}} + \sum k t_k \frac{\partial}{\partial t_{k+n}} + \sum_{a=1}^{n-1}\frac{\partial^2}{\partial t_a \partial t_{n-a}} + 2N\frac{\partial}{\partial t_n} + N^2\delta_{n,0}\right) Z_N\{t\} = 0, \quad n \ge -1 \tag{3}$$

$$\left(-\mu\frac{\partial}{\partial t_{n+1}} + \sum_k k t_k \frac{\partial}{\partial t_{k+n}} + \sum_{a=1}^{n-1}\frac{\partial^2}{\partial t_a \partial t_{n-a}} + (N_1+N_2)\cdot(1-\delta_{n,0})\frac{\partial}{\partial t_n} + N_1N_2\cdot\delta_{n,0}\right) Z_{N_1,N_2}\{t\} = 0 \quad n \ge 0 \tag{4}$$

This was already a lot and made these "matrix model" $\tau$-functions the most important new special functions for the modern epoch

*E-mail addresses:* mironov@lpi.ru, mironov@itep.ru (A. Mironov), morozov@itep.ru (A. Morozov).

http://dx.doi.org/10.1016/j.physletb.2017.05.094

of string theory calculations [3,9]. The comprehensive theory of genus expansion, the AMM/EO topological recursion [10], a non-perturbative treatment [11] of the Dijkgraaf-Vafa phases [11–17] and the check-operators [15] provided a dream-like standards for non-perturbative calculations and were already applied in a variety of topics ranging from knot theory [18,19] to wall-crossing [20] and AGT relations [21].

However, a long-standing problem overshadowed all these successes: they resulted from an artful combination and interplay of the two features: the Virasoro constraints and integrability. These two are intimately related: from the early days it was realized [5,22] that integrability makes the entire infinite set of Virasoro relations into a corollary of the lowest one named “string equation”. From this perspective, integrability describes the partition function in terms of the Grassmannian, and the string equation picks up a concrete point in the Grassmannian. This was a very big conceptual achievement, still its power is limited: to get an answer to any concrete question, which in quantum field theory is a calculation of the concrete correlation function, one still needs to do a complicated work growing with the complexity of the correlator.

Worse of all, while the theory was nicely developing in the direction of non-perturbative calculus, in the background of non-vanishing coupling $t_k = T_k$, the Gaussian averages at $t = 0$ remained untamed. One sometimes manages to artfully deduce a particular formula for a class of correlators like the famous Harer-Zagier answer [11,23] for the arbitrary Gaussian average

$$\sum_{k=0}^{\infty}\frac{z^k}{(2k-1)!!}\Big\langle \mathrm{Tr}_{N\times N} M^{2k}\Big\rangle = \frac{\mu}{2z}\left(\left(\frac{\mu+z}{\mu-z}\right)^N - 1\right) \qquad (5)$$

for any single-trace operator in the Hermitian model (see [24] on generalizations to two- and three-trace case based partly on related achievements in [?]). Nevertheless, it remains more a piece of art than a theory capable of producing answers to arbitrary questions.

A slightly better result was a $W$-representation [26], where the partition function is represented as an action of integrability-preserving operator $\hat W$ on a trivial $\tau$-function, e.g.

$$Z_N\{t\} = e^{\frac{1}{2\mu}\hat W\{t\}}\cdot e^{Nt_0} \qquad (6)$$

with

$$\hat W_2 = \sum_{a,b}\left(abt_at_b\frac{\partial}{\partial t_{a+b-2}} + (a+b+2)t_{a+b+2}\frac{\partial^2}{\partial t_a\partial t_b}\right) \qquad (7)$$

and

$$Z_{N_1\times N_2}\{t\} = \exp\left\{\frac{1}{\mu}\Big(N_1N_2t_1 + (N_1+N_2)\hat L_1 + \hat W_1\Big)\right\}\cdot 1 \qquad (8)$$

with

$$\hat L_1 = \sum_m (m+1)t_{m+1}\frac{\partial}{\partial t_m}, \qquad (9)$$

$$\hat W_1 = \sum_{a,b} abt_at_b\frac{\partial}{\partial t_{a+b-1}} + (a+b+1)t_{a+b+1}\frac{\partial^2}{\partial t_a\partial t_b} \qquad (10)$$

Since the operators $\hat W$, $\hat L$ have non-vanishing grading 2 and 1, each correlator of a given grading appears in just *one* term of the expansion of the exponential and thus becomes a solution to a finite-dimensional problem: that of iterative action of finite number of $\hat W$'s on $e^{Nt_0}$ and of $\hat W$ and $\hat L$'s on unity. Still, no explicit formula for a generic correlator was so far obtained in this way.

However, as a byproduct of recent studies of far more complicated problems in [1] and [2], we introduced a set of formulas, some of them being probably new, which do provide a *full solution* of the Hermitian and rectangular complex models. Since those papers were not focused on these particular results, they remained not-so-well-noticed, while they certainly deserve a dedicated and absolutely clear presentation. Namely, the partition functions have very explicit character expansions and arbitrary Gaussian correlators are represented by finite sums. Formulas are much simpler for the rectangular complex model with a bigger symmetry (see earlier results in [27,28]):

$$\boxed{Z_{N_1\times N_2}\{t\} = \sum_R \mu^{-|R|}\frac{D_R(N_1)\,D_R(N_2)}{d_R}\cdot\chi_R\{t\}} \qquad (11)$$

$$\boxed{\mathcal{O}_\Lambda = \left\langle\prod_{i=1}^{l_\Delta}\mathrm{Tr}\,(MM^\dagger)^{l_i}\right\rangle_G = \frac{1}{\mu^{|\Lambda|}}\sum_{R\vdash|\Lambda|}\frac{D_R(N_1)\,D_R(N_2)}{d_R}\cdot\psi_R(\Lambda)} \qquad (12)$$

where $\Lambda = \{l_1 \ge l_2 \ldots \ge l_\Delta > 0\}$ and $R$ are the Young diagrams of the given size (number of boxes) $|\lambda| = \sum_i l_i$, and $D_R(N)$, $\chi_R\{t\}$, $\psi_R(\Lambda)$ and $d_R$ are respectively the dimension of representation $R$ for the linear group $GL(N)$, the linear character (Schur polynomial), the symmetric group character and the dimension of representation $R$ of the symmetric group $S_{|R|}$ divided by $|R|!$, $D_R(N) = \chi_R\{t_n = N/n\}$, $d_R = \chi_R\{t_n = \delta_{n,1}\}$ [29]. Further we will also need a factor $z_\Lambda$, which is the standard symmetric factor of the Young diagram (order of the automorphism) [29].

For the Hermitian model, the structure of the formulas is the same, but the Young diagrams are of even dimension (number of boxes) and additional weight factors emerge made from differently-normalized symmetric characters $\varphi_R(\Delta) = \psi_R(\Delta)\cdot d_R^{-1}\cdot z_\Lambda^{-1}$ from [30] (cf. also with a Fourier expansion of [31, Eq. (2.18)] in terms of characters):

$$\boxed{Z_N\{t\} = \sum_{R\text{ of even size}}\mu^{-|R|}\varphi_R\Big(\underbrace{[2,\ldots,2]}_{|R|/2}\Big)\cdot D_R(N)\cdot\chi_R\{t\}} \qquad (13)$$

$$\boxed{\mathcal{O}_\Lambda = \left\langle\prod_{i=1}^{l_\Delta}\mathrm{Tr}\,M^{l_i}\right\rangle_G = \frac{1}{\mu^{|\Lambda|}}\sum_{R\vdash|\Lambda|}\varphi_R\Big(\underbrace{[2,\ldots,2]}_{|R|/2}\Big)\cdot D_R(N)\cdot\psi_R(\Lambda)} \qquad (14)$$

All ingredients in these formulas are well known objects from the basic representation theory, the least trivial of them, $\psi_R(\Lambda)$ are called by the MAPLE command $Chi(R,\Lambda)$ in the *combinat* package. The strangely-looking factor in (14) is actually

$$\varphi_R\Big(\underbrace{[2,\ldots,2]}_{|R|/2}\Big) = \frac{1}{d_R}\chi_R\left\{t_n = \frac{1}{2}\delta_{n,2}\right\} \qquad (15)$$

*The four formulas (11)–(14) provide a complete solution to the models, much more powerful than just integrability or Virasoro constraints.*

These formulas (11), (13) once again emphasize a relative simplicity of the complex model of [7] with respect to the more familiar Hermitian one. For example, the very first check of integrability, the simplest Plücker relation

$$w_{[2,2]}w_{[0]} - w_{[2,1]}w_{[1]} + w_{[2]}w_{[1,1]} = 0 \qquad (16)$$

between the coefficients $w_R$ in the character expansion of the KP tau-function $\tau\{t\} = \sum_R w_R\chi_R\{t\}$ in the case of (13) is just an obvious identity between the dimensions

$$\frac{N_1^2(N_1^2-1)\,N_2(N_2^2-1)}{12} - \frac{N_1(N_1^2-1)\,N_2^2(N_2-1)}{3}\cdot N_1N_2 + \frac{N_1(N_1+1)\,N_2(N_2+1)}{2}\cdot\frac{N_1(N_1-1)\,N_2(N_2-1)}{2} = 0$$