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Vacuum polarization energy for general backgrounds in one space dimension

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ABSTRACT

For field theories in one time and one space dimensions we propose an efficient method to compute the vacuum polarization energy of static field configurations that do not allow a decomposition into symmetric and anti-symmetric channels. The method also applies to scenarios in which the masses of the quantum fluctuations at positive and negative spatial infinity are different. As an example we compute the vacuum polarization energy of the kink soliton in the ϕ^6 model. We link the dependence of this energy on the position of the soliton to the different masses.

1. Introduction

Vacuum polarization energies (VPE) sum the shifts of zero point energies of quantum fluctuations that interact with a (classical) background potential. Spectral methods [1] have been very successful in computing VPEs particularly for background configurations with sufficient symmetry to facilitate a partial wave decomposition for the quantum fluctuations. In this approach scattering data parameterize Green functions from which the VPE is determined. In particular the imaginary part of the two-point Green function at coincident points, *i.e.* the density of states, is related to the phase shift of potential scattering [2]. Among other features, the success of the spectral methods draws from the direct implementation of background independent renormalization conditions by identifying the Born series for the scattering data with the expansion of the VPE in the strength of the potential. The ultra-violet divergences are contained in the latter and can be re-expressed as regularized Feynman diagrams. In renormalizable theories the divergences are balanced by counterterms whose coefficients are fully determined in the perturbative sector of the quantum theory in which the potential is zero.

For field theories in one space dimension the partial wave decomposition separates channels that are even or odd under spatial reflection. We propose a very efficient method, that in fact is based on the spectral methods, to numerically compute the VPE for configurations that evade a decomposition into parity even and odd channels. This is particularly interesting for field theories that contain classical soliton solutions connecting vacua in which the masses of the quantum fluctuations differ. A prime example is the ϕ^6 model. For this model some analytical results, in particular the scattering data for the quantum fluctuations, have been discussed a while ago in Refs. [3,4]. However, a full calculation of the VPE has not yet been reported. A different approach, based on the heat kernel expansion with ζ-function regularization [5,6] has already been applied to this model [7].[1] This approach requires an intricate formalism on top of which approximations (truncation of the expansion) are required. We will see that they become less accurate as the background becomes sharper. We also note that a similar problem involving distinct vacua occurs in scalar electrodynamics when computing the quantum tension of domain walls [11].

We briefly review the setting of the one-dimensional problem. The dynamics of the field $\phi = \phi(t,x)$ is governed by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi\, \partial^\mu \phi - U(\phi)\,. \tag{1}$$

The self-interaction potential $U(\phi)$ typically has distinct minima and there may exist several static soliton solutions that interlink between two such minima as $x \to \pm\infty$. We pick a specific soliton, say $\phi_0(x)$ and consider small fluctuations about it

$$\phi(t,x) = \phi_0(x) + \eta(t,x)\,. \tag{2}$$

Up to linear order, the field equation turns into a Klein–Gordon type equation

$$\left[\partial_\mu \partial^\mu + V(x)\right]\eta(t,x) = 0\,, \tag{3}$$

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[1] See Refs. [8–10] for reviews of heat kernel and ζ-function methods.

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where $V(x) = U''(\phi_0(x))$ is the background potential generated by the soliton. At spatial infinity $V(x)$ approaches a constant to be identified as the mass (squared) of the quantum fluctuations. In general, as *e.g.* for the ϕ^6 model with $U(\phi) = \frac{\lambda}{2}\phi^2(\phi^2 - \Lambda^2)^2$, we allow $\lim_{x\to-\infty} V(x) \neq \lim_{x\to\infty} V(x)$. This gives rise to different types of quantum fluctuations. While ϕ_0 is classical, the fluctuations are subject to canonical quantization so that the above harmonic approximation yields the leading quantum correction. As a consequence of the interaction with the background the zero point energies of all modes change and the sum of all these changes is the VPE, *cf.* Sec. 3.

2. Phase shifts

As will be discussed in Sec. 3 the sum of the scattering (eigen)phase shifts is essential to compute the VPE from spectral methods. We extract scattering data from the stationary wave equation, $\eta(t,x) \to \mathrm{e}^{-iEt}\eta(x)$,

$$E^2\eta(x) = \left[-\partial_x^2 + V(x)\right]\eta(x)\,. \tag{4}$$

According to the above described scenario we define $m_L^2 = \lim_{x\to-\infty} V(x)$ and $m_R^2 = \lim_{x\to\infty} V(x)$ and take the convention $m_L \le m_R$, otherwise we just relabel $x \to -x$. We introduce a discontinuous pseudo potential

$$V_p(x) = V(x) - m_L^2 + \left(m_L^2 - m_R^2\right)\Theta(x_m) \tag{5}$$

with $\Theta(x)$ being the step function. Any finite value may be chosen for the matching point x_m. In contrast to $V(x)$, $V_p(x) \to 0$ as $x \to \pm\infty$. Then the stationary wave equation, (4) reads

$$\left[-\partial_x^2 + V_p(x)\right]\eta(x) = \begin{cases} k^2\eta(x)\,, & \text{for} \quad x \le x_m \\ q^2\eta(x)\,, & \text{for} \quad x \ge x_m \end{cases} \tag{6}$$

where $k = \sqrt{E^2 - m_L^2}$ and $q = \sqrt{E^2 - m_R^2} = \sqrt{k^2 + m_L^2 - m_R^2}$. We emphasize that solving Eq. (6) is equivalent to solving Eq. (4). We factorize coefficient functions $A(x)$ and $B(x)$ appropriate for the scattering problem via $\eta(x) = A(x)\mathrm{e}^{ikx}$ for $x \le x_m$ and $\eta(x) = B(x)\mathrm{e}^{iqx}$ for $x \ge x_m$:

$$A''(x) = -2ikA'(x) + V_p(x)A(x) \qquad \text{and}$$
$$B''(x) = -2iqB'(x) + V_p(x)B(x)\,, \tag{7}$$

where a prime denotes a derivative with respect to x. In appendix B of Ref. [2] related functions, $g_\pm(x)$ were introduced to parameterize the Jost solutions for imaginary momenta. The boundary conditions $A(-\infty) = B(\infty) = 1$ and $A'(-\infty) = B'(\infty) = 0$ yield the scattering matrix by matching the solutions at $x = x_m$. Above threshold, $k \ge \sqrt{m_R^2 - m_L^2}$ so that q is real, the scattering matrix is

$$S(k) = \begin{pmatrix} \mathrm{e}^{-iqx_m} & 0 \\ 0 & \mathrm{e}^{ikx_m} \end{pmatrix} \begin{pmatrix} B & -A^* \\ iqB + B' & ikA^* - A'^* \end{pmatrix}^{-1} \times \begin{pmatrix} A & -B^* \\ ikA + A' & iqB^* - B'^* \end{pmatrix} \begin{pmatrix} \mathrm{e}^{ikx_m} & 0 \\ 0 & \mathrm{e}^{-iqx_m} \end{pmatrix}, \tag{8}$$

where $A = A(x_m)$, etc. are the coefficient functions at the matching point. Conventions are that the diagonal and off-diagonal elements of S contain the transmission and reflections coefficients, respectively [12]. Below threshold we parameterize for $x \ge x_m$: $\eta(x) = B(x)\mathrm{e}^{-\kappa x}$ with $\kappa = \sqrt{m_R^2 - m_L^2 - k^2} \ge 0$ replacing $-iq$ in Eq. (7) so that $B(x)$ is real. Then

$$S(k) = -\frac{A\left(B'/B - \kappa - ik\right) - A'}{A^*\left(B'/B - \kappa + ik\right) - A'^*}\,\mathrm{e}^{2ikx_m} \tag{9}$$

is the reflection coefficient. In both cases we compute the sum of the eigenphase shifts $\delta(k) = -(i/2)\ln\det S(k)$. The negative sign on the right hand side of Eq. (9) suggests that (in most cases) $\delta(0)$ is an odd multiple of $\frac{\pi}{2}$ in agreement with Levinson's theorem. When the scattering problem diagonalizes into symmetric (S) and anti-symmetric (A) channels and taking $\delta(k) \to 0$ as $k \to \infty$, the theorem states that $\delta_S(0) = \pi(n_s - \frac{1}{2})$ and $\delta_A(0) = \pi n_A$, where n_S and n_A count the bound states in the two channels [13,14]. The additional $-\pi/2$ in the symmetric channel arises because in that channel it is the derivative of the wave function that vanishes at $x = 0$, rather than the wave function itself. For scattering off a background that does not decompose into these channels we have $\delta(0) = \pi(n - \frac{1}{2})$, where n is the total number of bound states [12]. There are particular cases in which $\delta(0)$ is indeed an integer multiple of π. Examples are reflectionless potentials and the case $V(x) \equiv 0$. Then there exist threshold states contributing $\frac{1}{2}$ to n.

The step potential of hight $m_R^2 - m_L^2$ centered at $x = x_m$ corresponds to $V_p \equiv 0$. In this case the wave equation is solved by $A(x) = B(x) \equiv 1$ and

$$\delta_{\rm step}(k) = \begin{cases} (k-q)x_m\,, & \text{for} \quad k \ge \sqrt{m_R^2 - m_L^2} \\ kx_m - \arctan\left(\frac{\sqrt{m_R^2 - m_L^2 - k^2}}{k}\right), & \text{for} \quad k \le \sqrt{m_R^2 - m_L^2} \end{cases} \tag{10}$$

agrees with textbook results.

3. Vacuum polarization energy

Formally the VPE is the sum of the shifts of the zero point energies due to the interaction with a background potential that is generated by the field configuration ϕ_0,

$$E_{\rm vac}[\phi_0] = \frac{1}{2}\sum_j \left(E_j[\phi_0] - E_j^{(0)}\right) + E_{\rm ct}[\phi_0]\,. \tag{11}$$

Regularization for this logarithmically divergent sum is understood. When combined with the counterterms, $E_{\rm ct}$ a unique finite result arises after removing regularization. Typically there are two contributions in the sum of Eq. (11): (i) explicit bound and (ii) continuous scattering states. The latter part is obtained as an integral over one particle energies weighted by the change in the density of states, $\Delta\rho(k)$. We find the density $\rho(k) = \frac{dN(k)}{dk}$ for scattering modes incident from negative infinity by discretizing $kL + \delta(k) = N(k)\pi$ where $\delta(k)$ is phase shift. Adopting the continuum limit $L \to \infty$ and subtracting the result from the non-interacting case yields the Krein formula [15],

$$\Delta\rho(k) = \rho(k) - \rho^{(0)}(k) = \frac{1}{\pi}\frac{d}{dk}\delta(k)\,. \tag{12}$$

The situation for modes incident from positive infinity is not as straightforward. Here we count levels (above threshold) by setting $qL + \delta(k) = N(k)\pi$. Since k is the label for the free states we get an additional contribution to the change in the density of states

$$\frac{L}{\pi}\frac{d}{dk}\left[q - k\right] = \frac{L}{\pi}\left[\frac{k}{\sqrt{k^2 + m_L^2 - m_R^2}} - 1\right] = \frac{L}{\pi}\left[\frac{\sqrt{E^2 - m_L^2}}{\sqrt{E^2 - m_R^2}} - 1\right]. \tag{13}$$