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Schwinger-type parametrization of open string worldsheets

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A R T I C L E I N F O

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A B S T R A C T

A parametrization of (super) moduli space near the corners corresponding to bosonic or Neveu–Schwarz open string degenerations is introduced for worldsheets of arbitrary topology. With this parametrization, Feynman graph polynomials arise as the $\alpha' \to 0$ limit of objects on moduli space. Furthermore, the integration measures of string theory take on a very simple and elegant form.

1. Introduction

A very useful parametrization of the moduli of multiloop Riemann surfaces is given by Schottky groups, which manifested themselves automatically in the earliest approaches to multiloop string amplitudes. In this letter we describe (in section 2) a scheme for co-ordinatizing the moduli space of orientable *open string* worldsheets, in which all $3g - 3 + n$ real moduli are realized as 'lengths' of plumbing fixtures. In section 3 we see how Feynman graphs with various distinct topologies arise as the $\alpha' \to 0$ limit of such worldsheets, given an appropriate mapping between dimensionless pinching parameters p_i and Schwinger parameters t_i. In section 4 we show how the construction can be extended to the Neveu–Schwarz sector of superstrings, and present the elegant form taken by the leading part of the string measure in the pinching moduli. The pinching moduli are 'canonical parameters' in the sense of section 6.3 of reference [1], so their use makes Berezin integration on supermoduli space unambiguous. Proofs omitted in this letter are to be provided in a forthcoming work [2].

2. The parametrization

We are interested in describing worldsheets near complete "open string" degenerations; in such regions of moduli space the worldsheets may be constructed from 3-punctured discs glued together with strips. The topologically distinct degenerations can be classified as cubic ribbon graphs (*i.e.* graphs with a fixed cyclic ordering of the three edges incident on each vertex). Given a (not necessarily planar) cubic ribbon graph, we want to find "pinching parameters" $\{p_i\}$, *i.e.* local coordinates on Schottky space such that taking $p_i \to 0$ gives the corner corresponding to that degeneration.

To achieve this, we will provide an algorithm for writing down the g Schottky group generators γ_i and the n positions of punctures x_j as functions of the pinching parameters for a given cubic ribbon graph.

The algorithm may be arrived at by considering transition functions on a surface obtained by gluing together 3-punctured discs with open-string plumbing fixtures. All transition functions will be composed of two fundamental ones: one that cycles between local coordinates around the three punctures on a disc, and one which moves from one end of a plumbing fixture to the other.

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Let us consider first of all a 3-punctured disc. Let the punctures be labelled a_1, a_2 and a_3 with a clockwise ordering. We will need three local coordinate charts z_1, z_2, z_3 which vanish at their respective punctures; $z_i(a_i) = 0$. The upper-half-plane is the image of the disc under z_i and its boundary is mapped onto the projective real line. Let us also specify

$$z_i(a_{i+1}) = \infty \; ; \qquad z_i(a_{i-1}) = 1 \; , \tag{1}$$

where the indices are mod 3. Then there is a unique Möbius map ρ which acts as a **transition function cycling the three charts**. We want to have $z_i = \rho(z_{i+1})$, then we need

$$\rho(0) = \infty \; ; \qquad \rho(\infty) = 1 \; ; \qquad \rho(1) = 0 \; . \tag{2}$$

This is given by $\rho(z) = 1 - 1/z$, or as a matrix acting on the homogeneous coordinates,

$$\rho = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} , \tag{3}$$

which, of course, satisfies $\rho^3 = \mathrm{Id}$. So in general, on a 3-punctured disc the transition functions between these canonical charts are given by

$$\begin{cases} \rho & \leftrightarrow \text{move anticlockwise around the disc} \\ \rho^{-1} & \leftrightarrow \text{move clockwise around the disc.} \end{cases} \tag{4}$$

The other ingredient is the **open string plumbing fixture**. Suppose our surface includes two charts z, w whose images are contained in the upper-half-plane and include semi-discs of radius 1 centered on 0. Then if we fix a "pinching parameter" p with $0 < p < 1$ and cut out the semi-discs $|z| < p$, $|w| < p$ we can impose the equation

$$z\,w = -p \; , \tag{5}$$

for $|z| < 1$, $|w| < 1$, which we call an open string plumbing fixture between the two charts. Topologically, the effect is to attach a strip to the boundary of the surface, either adding a 'handle' or joining two previously disconnected components. When we take $p \to 0$ the strip degenerates leaving a node joining $z^{-1}(0)$ to $w^{-1}(0)$.

We can view the plumbing fixture as a transition function from the chart at one end to the chart at the other: let us define a Möbius map σ_p such that Eq. (5) can be written as $w = \sigma_p(z)$, *i.e.* $\sigma_p(z) \equiv -p/z$, or as a matrix

$$\sigma_p = \frac{1}{\sqrt{p}} \begin{pmatrix} 0 & -p \\ 1 & 0 \end{pmatrix} . \tag{6}$$

We can summarize its use as

$$\sigma_p \leftrightarrow \text{ traverse a plumbing fixture with pinching parameter } p \; . \tag{7}$$

Now let us consider a cubic ribbon graph Γ. Let us assign *three* coordinate charts to each vertex, with one associated to each incident half-edge. We can write down a sequence composed of the following two moves taking us from one chart to any other one:

- Moving (anti)clockwise between two charts associated to different half-edges incident at the same vertex.
- Moving from a chart associated to a half-edge of an internal edge E_k to a chart associated to its half-edge at the vertex at the other end.

It's crucial that an internal edge not be traversed before first moving onto the chart associated to its half-edge.

A sequence of such moves can be translated into a transition function with the following dictionary:

$$\begin{cases} \text{move anticlockwise around a vertex} & \leftrightarrow \rho \\ \text{move clockwise around a vertex} & \leftrightarrow \rho^{-1} \\ \text{traverse } E_k & \leftrightarrow \sigma_{p_k} \equiv \sigma_k \, , \end{cases} \tag{8}$$

where we have associated a pinching parameter p_k to every internal edge E_k.

Note that for multiply-connected graphs, this procedure gives multiple, distinct transition functions from one chart to another, since there are multiple paths between each pair of charts and each path gives a different transition function. This is because there is a Schottky group: each transition function is well-defined modulo the group action.

To be more explicit, let us pick a **"base chart"** z (*i.e.* a choice of one of the vertices in Γ *and* one of its incident half-edges). If the surface has g loops, then we can find g homologically independent closed paths P_i starting and ending at z. For each closed path, we can use Eq. (8) to write down a Möbius map; these g Möbius maps are the Schottky group generators γ_i.

Furthermore, suppose Γ has n external edges (corresponding to punctures in the surface). Each external edge has a coordinate chart, in which the punctures are at 0. We can write down paths P_j from these charts back to the base chart z, and again using the dictionary Eq. (8), we can find Möbius maps V_j which are transition functions from these charts to the base chart z. Then the positions of the punctures as seen in the base chart will be given as

$$x_j = V_j(0) \, . \tag{9}$$

So we have defined a Riemann surface by a set of transition functions which depend on a set of parameters $\{p_i\}$. The number of parameters equals the number of internal edges, which by elementary graph topology is $3g - 3 + n$, coinciding with the real dimension