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Horizon thermodynamics in fourth-order gravity

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ABSTRACT

In the framework of horizon thermodynamics, the field equations of Einstein gravity and some other second-order gravities can be rewritten as the thermodynamic identity: $dE = TdS - PdV$. However, in order to construct the horizon thermodynamics in higher-order gravity, we have to simplify the field equations firstly. In this paper, we study the fourth-order gravity and convert it to second-order gravity via a so-called "Legendre transformation" at the cost of introducing two other fields besides the metric field. With this simplified theory, we implement the conventional procedure in the construction of the horizon thermodynamics in 3 and 4 dimensional spacetime. We find that the field equations in the fourth-order gravity can also be written as the thermodynamic identity. Moreover, we can use this approach to derive the same black hole mass as that by other methods.

1. Introduction

It has long been known that gravitational system has thermodynamic properties since the works of Hawking and Bekenstein [1,2]. Just like conventional thermodynamic systems, black holes also have the temperature, entropy and other thermodynamic quantities. Besides, black holes also have fruitful phase structures [3–16]. Not only that, it was found that the field equations of Einstein gravity and other more general gravitational theories, such as $f(R)$ gravity, can be derived from an equation of state of local spacetime thermodynamics [17,18].

There is another route to explore the relationship between the gravitational system and its relevant thermodynamic properties. It is the framework of horizon thermodynamics proposed by Padmanabhan [19]. It is shown that Einstein's field equations for a spherically symmetric spacetime can be written in the form of thermodynamic identity: $dE = TdS - PdV$. This makes the connection between gravity and thermodynamics more closely. The radial pressure P is the $\binom{r}{r}$ component of energy–momentum tensor. This approach has also been extended to the non-spherically symmetric cases [20,21] and other theories of gravity, such as Lovelock gravity [22], Hořava–Lifshitz theory [23] and Einstein gravity with conformal anomaly [24].

However, we can notice that many previous works on horizon thermodynamics were based on the second-order gravities. This means that in the field equations there are at most the second-order derivatives of metric functions. In fact, one can find that in many cases only the first-order derivatives of metric functions exist. The work [23] on the Hořava–Lifshitz theory is the first study on horizon thermodynamics in higher-order derivative gravity. But it is shown that the field equations of Hořava–Lifshitz gravity in the static, spherically symmetric case, only include the first-order derivatives of metric functions. Generally, in higher derivative gravities, the field equations are full of higher-order derivatives of metric functions and are very complicated. We cannot directly extend the previous approach to these theories. We should first reduce the higher-derivative gravity to some lower-derivative gravity. This process can be done via a "Legendre" transformation [25–27]. One can even convert the higher-derivative gravity from the original Jordan frame into Einstein frame by a conformal transformation [28,29] or field redefinition [30]. Under some conditions, one can verify the equivalence of black hole thermodynamics between the two frames [30,31]. However, we do not want to deal with the horizon thermodynamics of higher-derivative gravity in the Einstein frame, because there is still no consensus on the physical equivalence between the Jordan frame and the Einstein frame [32–34]. In this paper we will study a fourth-order derivative gravity. Via the "Legendre" transformation, it can be reduced to a second-order derivative gravity with some additional auxiliary fields, which is still equivalent to the original fourth-order derivative gravity. In this way, the field equations can be simplified greatly. Thus, we

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can extract the useful information from the field equations to construct the horizon thermodynamics.

The plan of this paper is as follows: In Sec. 2 we give a very short introduction to horizon thermodynamics in Einstein gravity. We present the necessary demonstrations on some notations. In Sec. 3 we introduce the fourth-derivative gravity theory and obtain the second-derivative gravity via the "Legendre" transformation. In Sec. 4 we give some examples to show the horizon thermodynamics in fourth-derivative gravity in 3 and 4 dimensional spacetime. In Sec. 5 we summarize our results and discuss the possible future directions. In Appendix, we give the complete form of some field equations in components.

2. Horizon thermodynamics in Einstein gravity

In this section, we simply introduce the horizon thermodynamics in Einstein gravity first proposed in [19]. For a static, spherically symmetric spacetime, the metric in the Schwarzschild gauge can be written as

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega^2. \tag{2.1}$$

Substituting the metric into Einstein field equation

$$G^{\mu}{}_{\nu} = R^{\mu}{}_{\nu} - \frac{1}{2}Rg^{\mu}{}_{\nu} = 8\pi T^{\mu}{}_{\nu}, \tag{2.2}$$

one can obtain

$$r_+ f'(r_+) - 1 = 8\pi r_+^2 P, \tag{2.3}$$

and thus

$$d\left(\frac{r_+}{2}\right) = \frac{f'(r_+)}{4\pi} d(\pi r_+^2) - PdV, \tag{2.4}$$

where r_+ represents the position of the event horizon, which must satisfy $f(r_+) = 0$. $P = T^r{}_r|_{r=r_+}$, is the radial pressure of matter at the horizon. $V = 4\pi r_+^3/3$ is called the "areal volume". According to Eq. (2.1), it is just the volume of the black hole with horizon radius r_+ in the coordinate.

Considering the temperature of the black hole is

$$T = \frac{\kappa}{2\pi} = \frac{f'(r_+)}{4\pi}, \tag{2.5}$$

Eq. (2.4) is just the conventional thermodynamic identity $dE = TdS - PdV$ with $E = r_+/2$, $S = A/4 = \pi r_+^2$. In the source-free case, the metric function represents Schwarzschild black hole. For this black hole, E is just the mass M of the black hole.

This result above only depends on the theories of gravity under consideration. It has nothing to do with the concrete black hole solution. The contributions from matter fields have been contained in the pressure P. Obviously, except for vacuum cases, $E \neq M$ generally.

In this case, only two pairs of thermodynamic variables exist, which are the intensive quantities $(T,\ P)$ and the extensive quantities $(S,\ V)$. In this framework the thermodynamic properties are directly related to the gravitational theories under consideration. The details of matter content are not important and the concrete black hole solutions are also not necessary. In this framework, we have studied the phase transitions and thermodynamic stabilities of black holes in general relativity and Gauss–Bonnet gravity [35].

3. The fourth-order gravity

In this section we will generalize the original horizon thermodynamics approach to the higher-derivative gravity. Let us consider a fourth-derivative gravity action

$$\begin{aligned} S = S_G + S_M &= \int d^d x\, (\mathcal{L}_G + \mathcal{L}_M) \\ &= \int d^d x \sqrt{-g}\, (L_G + L_M)\,, \end{aligned} \tag{3.1}$$

where S_M represents the action of matter fields, and the gravitational Lagrangian L_G takes the form

$$L_G = \frac{1}{\kappa}\left(R - 2\Lambda + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu}\right), \tag{3.2}$$

with $\kappa = 16\pi G_d$. It should be noted that the matter fields are necessary to derive the horizon thermodynamics, although its concrete form is not necessary. We need energy–momentum tensor to determine the PdV term uniquely.

The field equations that follow from the action Eq. (3.1) are

$$\mathcal{G}_{\mu\nu} + E_{\mu\nu} = 8\pi G_d T_{\mu\nu}, \tag{3.3}$$

where

$$\begin{aligned} G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu}, \\ E_{\mu\nu} &= 2\beta(R_{\mu\rho}R_{\nu}{}^{\rho} - R^{\rho\sigma}R_{\rho\sigma}\, g_{\mu\nu}) + 2\alpha R\,(R_{\mu\nu} - R\, g_{\mu\nu}) \\ &\quad + \beta\,(\Box R_{\mu\nu} + \nabla_\rho \nabla_\sigma R^{\rho\sigma}\, g_{\mu\nu} - 2\nabla_\rho \nabla_{(\mu} R_{\nu)}{}^{\rho}) \\ &\quad + 2\alpha\,(g_{\mu\nu}\Box R - \nabla_\mu \nabla_\nu R). \end{aligned} \tag{3.4}$$

If substituting the metric (2.1) into these field equations, the expressions are so complicated that one cannot directly construct the horizon thermodynamics.

Now we employ the "Legendre" transformation to simplify the field equations. We can introduce two conjugate fields in the following way:

$$\Phi = \frac{\delta \mathcal{L}}{\delta R}, \quad \Psi^{\mu\nu} = \frac{\delta \mathcal{L}}{\delta R_{\mu\nu}}, \tag{3.5}$$

where $\mathcal{L} = \mathcal{L}_G + \mathcal{L}_M$.

We can further set $\sqrt{-g}\phi = \Phi$ and $\sqrt{-g}\psi^{\mu\nu} = \Psi^{\mu\nu}$. In this way, we can obtain

$$\phi = \frac{1}{\kappa}(1 + 2\alpha R), \quad \psi^{\mu\nu} = \frac{2\beta}{\kappa}R^{\mu\nu}. \tag{3.6}$$

Now we take the "Legendre" transformation according to the two pairs of conjugated quantities. First, we should invert Eq. (3.6) to obtain R and $R_{\mu\nu}$ as functions of ϕ and $\psi_{\mu\nu}$, respectively. This can be easily done. Then substituting them into the following definition:

$$\begin{aligned} \mathcal{H}(\phi, \psi_{\mu\nu}) &= \Phi R + \Psi^{\mu\nu}R_{\mu\nu} - \mathcal{L} \\ &= \sqrt{-g}\left[\frac{(\kappa\phi - 1)^2}{4\alpha\kappa} + \frac{\kappa}{4\beta}\psi^{\mu\nu}\psi_{\mu\nu} + \frac{2\Lambda}{\kappa} - L_M\right]. \end{aligned} \tag{3.7}$$

At last, we define

$$\begin{aligned} \mathcal{L}_H(g_{\mu\nu}, \phi, \psi_{\mu\nu}) &= \Phi R + \Psi^{\mu\nu}R_{\mu\nu} - \mathcal{H}(\phi, \psi_{\mu\nu}) \\ &= \sqrt{-g}\left[\phi R + \psi^{\mu\nu}R_{\mu\nu} - \frac{2\Lambda}{\kappa} - \frac{(\kappa\phi - 1)^2}{4\alpha\kappa} \right. \\ &\qquad \left. - \frac{\kappa}{4\beta}\psi^{\mu\nu}\psi_{\mu\nu} + L_M\right]. \end{aligned} \tag{3.8}$$

Treating $g_{\mu\nu}, \phi, \psi_{\mu\nu}$ as three independent field variables, the variation of the Lagrangian (3.8) yields the following field equations:

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