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# Gravitational surface Hamiltonian and entropy quantization

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ABSTRACT

The surface Hamiltonian corresponding to the surface part of a gravitational action has $xp$ structure where $p$ is conjugate momentum of $x$. Moreover, it leads to $TS$ on the horizon of a black hole. Here $T$ and $S$ are temperature and entropy of the horizon. Imposing the hermiticity condition we quantize this Hamiltonian. This leads to an equidistant spectrum of its eigenvalues. Using this we show that the entropy of the horizon is quantized. This analysis holds for any order of Lanczos–Lovelock gravity. For general relativity, the area spectrum is consistent with Bekenstein's observation. This provides a more robust confirmation of this earlier result as the calculation is based on the direct quantization of the Hamiltonian in the sense of usual quantum mechanics.



## 1. Introduction

Quantum mechanically, black holes are thermodynamic objects which have temperature [1] as well as entropy [2]. The exact expressions for these entities have been obtained by semi-classical treatment. In the absence of a true "quantum gravity" theory, the precise microscopic origin of entropy is unknown. Since there is no such complete theory,[1] we can only do semi-classical computations. One of the earlier attempts to describe these microstates, which are responsible for horizon entropy, originated from the seminal works of Bekenstein [2]. He showed that when a neutral particle is swallowed by a Kerr black hole, the lower bound on the increment of the horizon area is

$$(\Delta A)_{\rm min} = 8\pi l_p^2 \,, \tag{1}$$

when the particle has a finite size – not smaller than Compton wavelength. Here $l_p$ is the Planck length. After Bekenstein, the same was calculated for the assimilation of a charged particle [5] and it was found to have a similar Planck length square nature, but with a different numerical factor.

Even now, people are still trying to understand and find the implications of this result. One of the most important consequences is – it leads to the quantization of horizon area. A much more sophisticated calculation by Bekenstein and Mukhanov [6] yielded the area spectrum as $4l_p^2 \ln k$ with $k = 2$. An exactly identical expression was later derived by using quasinormal modes, except with $k = 3$ [7,8]. The importance of this result are as follows. It was found that this is consistent with the Gibbs' paradox [9] and also that the value of Immirzi parameter, in the context of Loop quantum gravity, can be fixed [10]. But unfortunately the derivation by quasinormal modes encountered a problem. Maggiore [11] observed that the imaginary part of the ringing frequency dominates compared to the real part in higher $n$ limit where $n$ is an integer. Now since the whole calculation is semi-classical, which is reliable at large $n$, one should take the imaginary part in the computation. Then this leads to the old result by Bekenstein: quantum of area is $8\pi l_p^2$. Consequently, several attempts [12–23] have been made to find the spectrum of area or entropy. It turned out that the quantum of entropy is much more natural than that of area [15,20]. In all cases, for Einstein's gravity, one finds that the spacing is given by (1).

Now it is quite evident that the semi-classical calculation is mostly in favor of Eq. (1). In this paper we make another attempt to quantize black hole entropy and area. The basic idea follows from an earlier result by one of the authors [24]. It has been observed that the surface part of the Einstein–Hilbert action has a structure like $xp$ where $x$ and $p$ are coordinate and conjugate momentum, respectively. Also the evaluation of the surface term on the horizon leads to the surface Hamiltonian which is the product of entropy and temperature. Moreover, one can identify that

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[1] Of course, there are some attempts like loop quantum gravity [3], string theory [4]. But none of them are self-sufficient.

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the entropy is equivalent to $x$ while the temperature plays the role of conjugate momentum $p$ (for details, see [24]). So everything reduces to a classical Hamiltonian $H = xp$. An extension to Lanczos–Lovelock gravity theory has also been done [25] and here also same conclusion was drawn.

In the present work, we discuss the quantization of $xp$ Hamiltonian. The standard quantization procedure leads to some confusions. The key point is that, the quantum Hamiltonian $\hat{H} = \hat{x}\hat{p}$ is not hermitian. So one might think that, a simple symmetric ordering of $\hat{H}$ will solve the problem. Unfortunately, there are few subtle points in this apparently simple program. As we shall see, self-adjoint extension of the Hamiltonian is necessary and this will naturally lead to boundary conditions on the wave function. Our main finding is that, in the context of a black hole, when boundary condition is coupled with first law of thermodynamics, it naturally gives area quantization. We show that the spacing is consistent with Bekenstein's old result. For GR it leads to Eq. (1).

Thus, we make a direct quantization of entropy/area, unlike the earlier attempts, in the sense that the surface Hamiltonian (product of horizon entropy and temperature for a black hole) is quantized in the language of usual quantum mechanics. Therefore our method is completely new and gives a direct evidence of quantization of entropy. The most interesting outcome is that the evaluated result matches with that from the earlier "indirect" calculations. Hence we reconfirm Bekenstein's original value of Eq. (1), obtained by semi-classical approach.

## 2. Surface Hamiltonian: a brief discussion

In this section, we shall briefly discuss about the surface part of the action of a gravitational theory so that a new reader can find the paper self-sufficient. Calculating this for a metric on the horizon, it will be shown that it has a thermodynamic interpretation. From there the surface Hamiltonian will be identified. Moreover, we shall discuss why such a Hamiltonian has $xp$ structure. Both the GR as well as more general theory like Lanczos–Lovelock gravity will be our attention. Here a summary of the required information, for clarity, will be introduced without any detailed calculation. An interested reader can discuss with the relevant references (e.g. [24,25]) for explicit analysis.

Very recently it has been shown that the thermodynamic structure of the gravitational theories can be discussed in terms of two variables. For GR these are given by $f^{ab}$ and $N^c_{ab}$ which are related to the usual variables by the following relations:

$$f^{ab} = \sqrt{-g}\, g^{ab}\ ;$$
$$N^c_{ab} = -\Gamma^c_{ab} + \frac{1}{2}\Big(\Gamma^d_{ad}\delta^c_b + \Gamma^d_{bd}\delta^c_a\Big)\ . \tag{2}$$

Most notable point about these variables is that $N^c_{ab}$ is the conjugate momentum of $f^{ab}$. Before going into the main idea, let us review some salient points. It is well known that, the Einstein–Hilbert (EH) Lagrangian *i.e.* $\sqrt{-g}R$ can be divided into two parts: one is quadratic in $\Gamma^a_{bc}$ and the other one is a total derivative part, we call them as bulk and surface parts respectively. Although these are not scalars individually, there are some important features associated with them. The bulk part alone gives Einstein's equations of motion without suffering any inconsistency as one does not need to impose vanishing of the both variation of metric and derivative of metric at the boundary; here one just imposes the vanishing of $\delta g^{ab}$ at the boundary (see p. 242 of [26]). On the other hand, the surface part is given by

$$\mathcal{L}_{\rm sur} = \frac{2}{16\pi G}\partial_c\Big[Q_a{}^{bcd}\Gamma^a_{bd}\Big]\ , \tag{3}$$

where $Q_a{}^{bcd} = 1/2(\delta^c_a g^{bd} - \delta^d_a g^{bc})$. This term when calculated on the $r =$ constant surface for a static black hole, the action gives entropy in the near horizon limit. In this case the time integration is taken within the periodicity of Euclidean time (see p. 663 of [26] for details). Of course, if one computes the total EH action for a spherically symmetric static metric it has a thermodynamical structure like $S - E/T$ where $E$ is identifies as the black hole energy [27]. Moreover the surface part of the gravitational action can be expressed as

$$\mathcal{A}_{\rm sur} = -\frac{1}{16\pi G}\int d^4x\,\partial_c\Big(f^{ab}N^c_{ab}\Big)\ . \tag{4}$$

Therefore the surface Lagrangian has a structure like $\partial(xp)$ with $x \equiv f^{ab}$ and $p \equiv N^c_{ab}$.

Now the calculation of this action on the null surface for static spacetime leads to

$$\mathcal{A}_{\rm sur} = -\frac{1}{16\pi G}\int dt d^2x_\perp n_c f^{ab}N^c_{ab} = -\int dt\,TS \tag{5}$$

where $n_c$ is the normal to the surface, $x_\perp$ refers to the transverse coordinates and $T = \hbar\kappa/2\pi$ and $S = A/4G\hbar = A/4l_p^2$ are the horizon temperature and entropy, respectively with $\kappa$ being the surface gravity and $A$ is the area of the horizon. Therefore the surface Hamiltonian is identified as $H_{\rm sur} = -\partial\mathcal{A}_{\rm sur}/\partial t = (1/16\pi G)\int d^2x_\perp n_c f^{ab}N^c_{ab} = TS$.[2] *From this we can immediately realize that the present Hamiltonian has xp structure*. More precisely, Hamiltonian density (Hamiltonian per unit transverse area), which is temperature times entropy density (entropy per unit transverse area) has this structure. Moreover, among the two thermodynamical variables ($T$ and $S$), one of them plays the role of $x$ while the other one is $p$. Now to properly identify these variables we can take the help of the following analysis. It has also been observed that if we take a variation of the Hamiltonian; i.e. $\delta H_{\rm sur} = 1/16\pi G\Big[(\delta f^{ab})(n_c N^c_{ab}) + (f^{ab})\delta(n_c N^c_{ab})\Big]$ and calculate them on the horizon, then these two parts lead to

$$\frac{1}{16\pi G}\int d^2x_\perp(\delta f^{ab})(n_c N^c_{ab}) = T\delta S\ ;$$
$$\frac{1}{16\pi G}\int d^2x_\perp(f^{ab})\delta(n_c N^c_{ab}) = S\delta T\ . \tag{6}$$

Therefore one can say that $S \equiv x$ while $T$ is the conjugate momentum of $S$; i.e. $T \equiv p$. The details of these discussion can be followed from [24]. Now since the action is given by (5) and $T$, $S$ are conjugate variables, in classical mechanics we have the following Poisson's bracket:

$$\{S, T\}_{\rm PB} = 1\ . \tag{7}$$

It should be mentioned that this feature is not restricted to GR; rather this is much more general. The same has also been concluded for a general Lanczos–Lovelock gravity. For that we refer to [25] for the readers. In this general case the conjugate variables are $\tilde{f}^{ab} = f^{ab}$ and $\tilde{N}^c_{ab} = Q^{cd}_{ae}\Gamma^e_{bd} + Q^{cd}_{be}\Gamma^e_{ad}$ where $Q^{ab}_{cd} = (1/m)P^{ab}_{cd}$ with $P^{ab}_{cd} = \partial L_m/\partial R^{cd}_{ab}$ and $m$ is the order of the Lanczos–Lovelock Lagrangian $L_m$. The surface action is exactly in similar form with the un-tilde variables are replaced by tilde variables. Calculation of it on the horizon leads to identical results like the GR case with the entropy is properly defined in terms of the relevant component of $Q^{ab}_{cd}$. *Hence we infer that the structure of surface Hamiltonian*

[2] It may be mentioned that the same $TS$ Hamiltonian can also be obtained from the Gibbons–Hawking–York surface term [28], but it is not known if it has similar *xp* structure. Therefore we restrict our discussion within the non-covariant form of the surface part of the main gravitational action.