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Thermodynamic volume of cosmological solitons

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ABSTRACT

We present explicit expressions of the thermodynamic volume inside and outside the cosmological horizon of Eguchi–Hanson solitons in general odd dimensions. These quantities are calculable and well-defined regardless of whether or not the regularity condition for the soliton is imposed. For the inner case, we show that the reverse isoperimetric inequality is not satisfied for general values of the soliton parameter a, though a narrow range exists for which the inequality does hold. For the outer case, we find that the mass M_{out} satisfies the maximal mass conjecture and the volume is positive. We also show that, by requiring M_{out} to yield the mass of dS spacetime when the soliton parameter vanishes, the associated cosmological volume is always positive.

1. Introduction

Understanding black holes as thermodynamic systems is a subject that continues to yield new insights into gravitational physics, providing us with important clues as to the nature of quantum gravity. Asymptotically Anti de-Sitter (AdS) black holes have been of particular interest in recent years, in part because of their significance in various proposed gauge-gravity dualities, but also because they have been shown to exhibit thermodynamic behaviour analogous to that in everyday life, a subject known as Black Hole Chemistry [1].

In Black Hole Chemistry the cosmological constant Λ is regarded as a thermodynamic variable, extending the phase space of black hole thermodynamics [2]. The mass of the black hole can be understood as enthalpy [3] and the cosmological constant as pressure, with a conjugate thermodynamic volume V [3–18]. From this perspective one can show that the celebrated Hawking–Page phase transition [19] can be understood as being analogous to a solid/liquid phase transition [1], and more generally that the 4-dimensional Reissner–Nordström AdS black hole can be interpreted as a Van der Waals fluid with the same critical exponents [13]. Along with more general Van der Waals behaviour with standard critical exponents [14,20–36], other 'chemical' black hole behaviour was subsequently discovered, such as reentrant phase transitions [37], tricritical points [38], Carnot cycles [39], isolated critical points [40,41], extensions to black rings [42], and superfluidity [43].

The role of the thermodynamic volume V is not yet fully understood. It was originally conjectured to satisfy a relation known as the *Reverse Isoperimetric Inequality* [42,44], which states that the isoperimetric ratio

$$\mathcal{R} = \left(\frac{(D-1)V}{\omega_{D-2}}\right)^{\frac{1}{D-1}} \left(\frac{\omega_{D-2}}{A}\right)^{\frac{1}{D-2}} \tag{1.1}$$

always satisfies $\mathcal{R} \geq 1$, where A is the horizon area, and ω_d stands for the area of the space orthogonal to constant (t, r) surfaces. Physically it implies, for example, that the black hole of given "volume" V with maximal entropy is the Schwarzschild–AdS black hole. However a class of black holes has recently been found that violates this conjecture [45,46], necessitating further investigation of the role and meaning of the volume [47]. The relationship of V to other proposed notions of volume [48,49] is an ongoing subject of investigation.

Our knowledge of thermodynamic volume, and more generally the thermodynamic behaviour of asymptotically de Sitter (dS) black holes, for which $\Lambda > 0$, is significantly more sparse [47, 50–55]. Yet their importance to cosmology and to a posited duality between gravity in de Sitter space and conformal field theory [56] make them important objects of investigation. However this is a complex problem, since the absence of a Killing vector that is everywhere timelike outside the black hole horizon renders a good notion of the asymptotic mass questionable. Furthermore, the presence of both a black hole horizon and a cosmological horizon yields two distinct temperatures, suggesting that the system is

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in a non-equilibrium state. This in turn leads to some ambiguity in interpreting the thermodynamic volume, since distinct volumes can be associated with each horizon. In all known examples the reverse isoperimetric inequality $\mathcal{R} \geq 1$ holds separately for each; however if the volume is taken to be the naive geometric volume in between these horizons then the isoperimetric inequality holds [57].

It would be preferable to study the 'chemistry' of cosmological horizons in isolation. For this we need a class of solutions that are not of constant curvature and that have only a cosmological horizon. Fortunately a broad class of such solutions exists: Eguchi–Hanson de Sitter solitons [58].

The Eguchi–Hanson (EH) metric is a self-dual solution of the four-dimensional vacuum Euclidean Einstein equations [59]. It has odd-dimensional generalizations that were discovered few years ago [58] in Einstein gravity with a cosmological constant. They are referred to as the Eguchi–Hanson solitons. For $\Lambda < 0$ they are horizonless solutions that in five dimensions are asymptotic to AdS_5/Z_p $(p \geq 3)$ and have Lorentzian signature, yielding a non-simply connected background manifold for the CFT boundary theory [60]. Solutions in higher dimensions have a more complicated asymptotic geometry. For $\Lambda > 0$ these solutions in any odd dimension have a single cosmological horizon, by which we mean that they have a Killing vector $\partial/\partial t$ that becomes spacelike at sufficiently large distance from the origin. Upon taking the mass to be the conserved quantity associated with this Killing vector at future infinity, and computing it using the counterterm method [61], these solutions all satisfy a *maximal mass conjecture* [62], whose implication is that they all have mass less than that of pure de Sitter spacetime with the same asymptotics.

In this paper we study the Eguchi–Hanson de Sitter (EHdS) solitons in the context of extended phase space thermodynamics. In this framework, we consider the cosmological constant as a thermodynamic variable equivalent to the pressure in the first law where

$$P = -\frac{\Lambda}{8\pi G} \tag{1.2}$$

though for $\Lambda > 0$ this quantity is negative and so is perhaps best referred to as a tension. Noting this, we shall continue to refer to P as pressure; its corresponding conjugate is the thermodynamic volume V and is defined from geometric arguments [57]. It ensures the validity of the extended first law

$$\delta M - T dS - V \delta P = 0 \tag{1.3}$$

and (consistent with Eulerian scaling) renders the Smarr formula valid:

$$(d-2)M - (d-1)TS + 2VP = 0 \tag{1.4}$$

where the spacetime dimension is given by $(d+1)$.

Motivated by the above, we use the Eguchi–Hanson solitons in de Sitter space to investigate their thermodynamics and cosmological volume in the context of extended phase space. The particular advantage afforded by these solutions is that, unlike the situation with de Sitter black holes, thermodynamic equilibrium is satisfied. We find explicit expressions for the thermodynamic volume inside and outside the cosmological horizon. For the inner case, the reverse isoperimetric inequality is satisfied only for a small range of $a > \sqrt{3/4}\ell$ when a regularity condition for the soliton is not satisfied. For the outer case, an important role is played by a Casimir-like term that appears as an arbitrary constant in the first law and Smarr relation. We compare our results to those obtained using the counterterm method [58] and we find that they match. We note that for this case the mass is always smaller than maximal mass given by the Casimir term and that the thermodynamic volume is always positive if the regularity condition is applied.

The outline of our paper is as follows: in the next section we introduce the EH Solitons in odd dimensions. We briefly discuss general considerations of their thermodynamics in section 3. In section 4, we make use of the first law and Smarr relation to compute the mass and thermodynamic volume of these solutions inside and outside the cosmological horizon of dS space. We show that explicit expressions for the two parameters can be found in general odd dimensions. We briefly summarize our results in a concluding section.

2. EHdS solitons

EHdS solitons [58,60] in general odd $(d+1)$ dimensions are exact solutions to the Einstein equations with $\Lambda > 0$, and have metrics of the form

$$ds^2 = -g(r)dt^2 + \left(\frac{2r}{d}\right)^2 f(r)\left[d\psi + \sum_{i=1}^{k}\cos(\theta_i)d\phi_i\right]^2 + \frac{dr^2}{g(r)f(r)} + \frac{r^2}{d}\sum_{i=1}^{k} d\Sigma^2_{2(i)} \tag{2.1}$$

in $d = 2k+2$ dimensions, where the metric functions are given by

$$g(r) = 1 - \frac{r^2}{\ell^2} \quad , \quad f(r) = 1 - \left(\frac{a}{r}\right)^d \tag{2.2}$$

with

$$d\Sigma^2_{2(i)} = d\theta_i^2 + \sin^2(\theta_i)d\phi_i^2 \tag{2.3}$$

and

$$\Lambda = +\frac{d(d-1)}{2\ell^2} \tag{2.4}$$

parametrizing the positive cosmological constant.

The radial coordinate $r \geq a$; for $r < a$ the metric changes signature, indicative of its solitonic character. There is a cosmological horizon at $r = \ell$. Constant (t, r) sections consist of the fibration of a circle over a product of k 2-spheres. Generalizations to Gauss–Bonnet gravity [63] and to spacetimes with more general base spaces [64] exist but we shall not consider these solutions here.

For $\ell \to \infty$, the metric (2.1) becomes

$$ds^2 = \left(\frac{2r}{d}\right)^2\left(1 - \left(\frac{a}{r}\right)^d\right)\left[d\psi + \sum_{i=1}^{k}\cos(\theta_i)d\phi_i\right]^2 + \frac{dr^2}{1-\left(\frac{a}{r}\right)^d} + \frac{r^2}{d}\sum_{i=1}^{k} d\Sigma^2_{2(i)} \tag{2.5}$$

for a constant $t =$ hypersurface. This class of metrics can be regarded as d-dimensional generalizations of the original [59] $d = 4$ Eguchi–Hanson metric.

In general, the metric (2.1) will not be regular unless some conditions are imposed to eliminate the singularities. Noting that a constant (t, r) section has the form

$$ds^2 = F(r)\left[d\psi + \sum_{i=1}^{k}\cos(\theta_i)d\phi_i\right]^2 + \frac{dr^2}{G(r)} \tag{2.6}$$

where $F(r) = \left(\frac{2r}{d}\right)^2 f(r)$ and $G(r) = f(r)g(r)$, regularity requires the absence of conical singularities. This implies that the periodicity of ψ at infinity must be an integer multiple of its periodicity as $r \to a$. Consequently