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Charged massive scalar field configurations supported by a spherically symmetric charged reflecting shell

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ABSTRACT

The physical properties of bound-state charged massive scalar field configurations linearly coupled to a spherically symmetric charged reflecting shell are studied *analytically*. To that end, we solve the Klein–Gordon wave equation for a static scalar field of proper mass μ, charge coupling constant q, and spherical harmonic index l in the background of a charged shell of radius R and electric charge Q. It is proved that the dimensionless inequality $\mu R < \sqrt{(qQ)^2 - (l+1/2)^2}$ provides an upper bound on the regime of existence of the composed charged-spherical-shell–charged-massive-scalar-field configurations. Interestingly, we explicitly show that the *discrete* spectrum of shell radii $\{R_n(\mu, qQ, l)\}_{n=0}^{n=\infty}$ which can support the static bound-state charged massive scalar field configurations can be determined analytically. We confirm our analytical results by numerical computations.

1. Introduction

The influential no-hair theorems [1] have revealed the interesting fact that spherically symmetric asymptotically flat black holes cannot support static massive scalar field configurations in their exterior regions [2–4]. Motivated by this well known property of spherically symmetric static black holes, we have recently [5] extended this no-scalar-hair theorem to the regime of regular [6] curved spacetimes. In particular, it was proved in [5] that spherically symmetric compact reflecting [7] stars cannot support regular self-gravitating neutral scalar field configurations in their exterior regions.

One naturally wonders whether this no-scalar-hair behavior [5] is a generic feature of compact reflecting objects? In particular, we raise here the following physically intriguing question: Can regular static *charged* massive scalar field configurations be supported by a compact spherically symmetric charged reflecting object? In order to address this interesting question, in this paper we shall study, using *analytical* techniques, the Klein–Gordon wave equation for a static linearized scalar field of proper mass μ and charge coupling constant q in the background of a spherically symmetric charged reflecting shell of radius R and electric charge Q.

Our results (to be proved below) reveal the fact that, for given parameters $\{\mu, q, l\}$ [8] of the charged massive scalar field, there exists a *discrete* set of shell radii $\{R_n(\mu, qQ, l)\}_{n=0}^{n=\infty}$ which can support the static bound-state charged massive scalar field configurations. In particular, as we shall explicitly show below, the regime of existence of these composed charged-spherical-shell–charged-massive-scalar-field configurations is restricted by the characteristic inequality $(qQ)^2 > (\mu R)^2 + (l+1/2)^2$ [9,10]. This relation implies, in particular, that spatially regular bound-state configurations made of neutral scalar fields [5] cannot be supported by a spherically symmetric compact reflecting object.

2. Description of the system

We shall analyze the physical properties of a scalar field Ψ of proper mass μ and charge coupling constant q which is linearly coupled to a spherically symmetric charged shell of radius R. The shell is assumed to have negligible self-gravity:

$$M, Q \ll R , \tag{1}$$

where $\{M, Q\}$ are the proper mass and electric charge of the shell, respectively.

Decomposing the static scalar field Ψ in the form [11]

$$\Psi(r,\theta,\phi) = \sum_{lm} e^{im\phi} S_{lm}(\theta) R_{lm}(r) , \tag{2}$$

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one finds that the spatial behavior of the radial scalar eigenfunctions $\{R_{lm}(r)\}$ in the spacetime region outside the charged spherical shell is governed by the ordinary differential equation [12–16]

$$\frac{d}{dr}\Big(r^2\frac{dR_{lm}}{dr}\Big)+UR_{lm}=0\ , \tag{3}$$

where

$$U=(qQ)^2-(\mu r)^2-K_l\ . \tag{4}$$

Here $K_l=l(l+1)$ is the characteristic eigenvalue of the angular scalar eigenfunctions $\{S_{lm}(\theta)\}$ [17,18].

The bound-state (spatially localized) charged massive scalar field configurations that we shall analyze below are characterized by asymptotically decaying eigenfunctions

$$\Psi(r\to\infty)\sim\frac{1}{r}e^{-\mu r} \tag{5}$$

at spatial infinity. In addition, the presence of the spherically symmetric charged reflecting shell at $r=R$ dictates the boundary condition

$$\Psi(r=R)=0 \tag{6}$$

for the characteristic scalar eigenfunctions.

In the next section we shall explicitly show that the radial differential equation (3), which determines the spatial behavior of the characteristic radial eigenfunctions $\{R_{lm}(r)\}$ of the charged massive scalar fields in the background of the charged spherical shell, is amenable to an *analytical* treatment.

3. The resonance equation for the composed charged-spherical-shell–charged-massive-scalar-field configurations

As we shall now show, the characteristic radial equation (3) for the charged massive scalar eigenfunction $R_{lm}(r)$ can be solved analytically. Defining the new radial function

$$\psi_{lm}=r^{1/2}R_{lm} \tag{7}$$

and using the dimensionless radial coordinate

$$z=\mu r\ , \tag{8}$$

one obtains the differential equation [19]

$$z^2\frac{d^2\psi}{dz^2}+z\frac{d\psi}{dz}-\big[z^2+(l+\frac{1}{2})^2-(qQ)^2\big]\psi=0 \tag{9}$$

for the characteristic radial scalar eigenfunction ψ.

The general solution of the radial differential equation (9) can be expressed in terms of the modified Bessel functions (see Eq. 9.6.1 of [17]) [20]:

$$\psi(z)=A\cdot K_\nu(z)+B\cdot I_\nu(z)\ , \tag{10}$$

where

$$\nu^2\equiv(l+\frac{1}{2})^2-(qQ)^2 \tag{11}$$

and $\{A,B\}$ are normalization constants. The asymptotic large-r (large-z) behavior of the radial solution (10) is given by (see Eqs. 9.7.1 and 9.7.2 of [17])

$$\psi(z\to\infty)=A\cdot\sqrt{\frac{\pi}{2z}}e^{-z}+B\cdot\frac{1}{\sqrt{2\pi z}}e^{z}\ . \tag{12}$$

Taking cognizance of the boundary condition (5), which characterizes the asymptotic spatial behavior of the bound-state (spatially localized) scalar configurations, one deduces that the coefficient of the exploding exponent in (12) must vanish:

$$B=0\ . \tag{13}$$

One therefore concludes that the bound-state configurations of the charged massive scalar fields in the background of the charged spherical shell are characterized by the radial eigenfunction

$$\psi(r)=A\cdot K_\nu(\mu r)\ . \tag{14}$$

Taking cognizance of Eq. (14) and the boundary condition (6) which is dictated by the presence of the spherically symmetric reflecting shell, one finds the characteristic resonance equation

$$K_\nu(\mu R)=0 \tag{15}$$

for the composed static charged-spherical-shell–charged-massive-scalar-field configurations. Interestingly, as we shall show below, the resonance condition (15) determines the *discrete* set of shell radii $\{R_n(\mu,qQ,l)\}_{n=0}^{n=\infty}$ which can support the bound-state charged massive scalar field configurations.

In the next section we shall prove that the resonance condition (15) can only be satisfied in the bounded regime

$$(qQ)^2>(\mu R)^2+(l+\frac{1}{2})^2\ . \tag{16}$$

The necessary inequality (16), to be proved below, implies in particular that spatially regular static bound-state configurations made of neutral scalar fields cannot be supported by a spherically symmetric compact reflecting object.

4. The domain of existence of the charged massive scalar hair

Using the boundary conditions (5) and (6), one concludes that the scalar eigenfunction ψ, which characterizes the radial behavior of the charged massive scalar fields, must have (at least) one extremum point, $z=z_{\text{peak}}$, outside the spherically symmetric charged reflecting shell. In particular, at this extremum point the radial scalar eigenfunction ψ is characterized by the relations

$$\{\frac{d\psi}{dz}=0\ \ \text{and}\ \ \psi\cdot\frac{d^2\psi}{dz^2}<0\}\quad\text{for}\quad z=z_{\text{peak}}\ . \tag{17}$$

Substituting (17) into (9), one finds the characteristic inequality

$$z^2_{\text{peak}}+(l+\frac{1}{2})^2-(qQ)^2<0\ . \tag{18}$$

Taking cognizance of (8) and using the inequality $r_{\text{peak}}>R$, one finds from (18) that the composed charged-spherical-shell–charged-massive-scalar-field configurations are characterized by the inequality (16). In particular, this inequality sets the upper bound

$$\mu R<\sqrt{(qQ)^2-(l+\frac{1}{2})^2} \tag{19}$$

on the radius of the central charged supporting shell.

For later purposes, it is important to point out that the inequality (19) [or equivalently, the inequality (16)] implies that the static charged massive scalar field configurations are characterized by the relation $\nu^2<0$ [see Eq. (11)], which implies

$$i\nu\in\mathbb{R}\ . \tag{20}$$