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Generalized sigma model with dynamical antisymplectic potential and non-Abelian de Rham's differential

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ABSTRACT

For topological sigma models, we propose that their local Lagrangian density is allowed to depend non-linearly on the de Rham's "velocities" DZ^A. Then, by differentiating the Lagrangian density with respect to the latter de Rham's "velocities", we define a "dynamical" anti-symplectic potential, in terms of which a "dynamical" anti-symplectic metric is defined, as well. We define the local and the functional antibracket via the dynamical anti-symplectic metric. Finally, we show that the generalized action of the sigma model satisfies the functional master equation, as required.

1. Introduction

When formulating a topological sigma model, one proceeds usually with an anti-symplectic configuration space $\{Z^A|\varepsilon(Z^A)=: \varepsilon_A\}$ whose anti-symplectic potential $V_A(Z), \varepsilon(V_A)=: \varepsilon_A+1$, is given originally [1–11]. At the same time, it is assumed usually that the kinetic part of the original local Lagrangian density $\mathcal{L}$ has the form $-(DZ^A)V_A(Z)$, where D is the de Rham's differential.

In the present paper, we generalize the local original Lagrangian density as to take the form $\mathcal{L}(Z, DZ)$ allowed to depend non-linearly on the de Rham's "velocities" DZ^A. Then, we define a "dynamical" anti-symplectic potential, $V_A(Z, DZ)$ as the derivatives of the new Lagrangian density with respect to the mentioned de Rham's "velocities". We define a local anti-symplectic metric in its covariant components $E_{AB}(Z, DZ)$ as the standard vorticity of the "dynamical" anti-symplectic potential $V_A(Z, DZ)$, in terms of explicit Z^A-derivatives.

We define both the local and the functional antibracket by the standard formulae via the "dynamical" anti-symplectic metric in its contravariant components $E^{AB}(Z, DZ)$. Finally, we show that the new action $\Sigma =: \int d\mu \mathcal{L}(Z, DZ)$ satisfies the functional master equation, provided the function $S(Z, DZ) =: \mathcal{L}(Z, DZ) + DZ^A V_A(Z, DZ)$ satisfies the local master equation.

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2. Non-Abelian de Rham's differential

Let Γ be an intrinsic configuration super-manifold,

$$\Gamma =: \{X^a, C^a | \varepsilon(X^a)=0,\ \varepsilon(C^a)=1, a=1,\dots,2m\}. \quad (2.1)$$

Let D be a non-Abelian de Rham's differential, as defined by the conditions

$$\varepsilon(D)=1, \quad D^2=\frac{1}{2}[D,D]=0, \quad D=-D^{\dagger}, \quad (2.2)$$

whose solution is sought for in the form,

$$D =: C^a \Lambda_a^b(X)\frac{\partial}{\partial X^b} + \frac{1}{2}C^b C^a \mathcal{U}^d_{ab}(X)\frac{\partial}{\partial C^d}, \quad (2.3)$$

with Λ_a^b being invertible, and U^d_{ab} being antisymmetric in its subscripts a, b [6]. The conditions (2.2) imply

$$\Lambda_a^c \partial_c \Lambda_b^d - (a \leftrightarrow b) = \mathcal{U}^c_{ab}\Lambda_c^d, \quad (2.4)$$

$$(-\Lambda_a^e \partial_e \mathcal{U}^d_{bc} + \mathcal{U}^e_{ab}\mathcal{U}^d_{ec}) + \text{cyclic perm.}(a,b,c) = 0. \quad (2.5)$$

The Jacobi relation (2.5) provides for the integrability of the Maurer–Cartan equation (2.4). In terms of the Boson integration measure,

$$d\mu(\Gamma) =: \rho(X)[dX][dC], \quad \rho =: \det(\Lambda^{-1}) = (\det(\Lambda))^{-1}, \quad (2.6)$$

the anti-Hermiticity of the differential D implies

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$$\rho^{-1}\partial_b(\rho\Lambda^b_a)+\mathcal{U}^b_{ab}=0. \tag{2.7}$$

In turn, it follows from (2.7),

$$\int d\mu(\Gamma)DF(\Gamma)=0,\quad F(\Gamma)\Big|_{\partial\Gamma}=0. \tag{2.8}$$

3. Generalized sigma model

Let Γ' be an antisymplectic phase space,

$$\Gamma'=:\{Z^A|\varepsilon(Z^A)=\varepsilon_A,\ A=1,\dots,2N\}, \tag{3.1}$$

with N being an equal number of Boson and Fermion variables among Z^A. Let us define the local action of the generalized sigma model in the form,

$$\Sigma=:\int d\mu(\Gamma)\mathcal{L}(Z(\Gamma),DZ(\Gamma)), \tag{3.2}$$

with the measure $d\mu(\Gamma)$ being defined in (2.6). A functional derivative of the action (3.2) has the form,

$$\frac{\delta\Sigma}{\delta Z^A(\Gamma)}=\partial_A\mathcal{L}+DV_A(-1)^{\varepsilon_A},\quad V_A=:-\frac{\partial\mathcal{L}}{\partial(DZ^A)}, \tag{3.3}$$

in terms of explicit Z^A-derivatives ∂_A. In turn, we have

$$\begin{aligned}DV_A(-1)^{\varepsilon_A}&=DZ^B\partial_BV_A(-1)^{\varepsilon_A}\\&=-\partial_BV_ADZ^B(-1)^{\varepsilon_B(\varepsilon_A+1)}.\end{aligned} \tag{3.4}$$

By inserting that into the relation (3.3), we get

$$\frac{\delta\Sigma}{\delta Z^A(\Gamma)}=E_{AB}DZ^B(-1)^{\varepsilon_B}+\partial_AS, \tag{3.5}$$

where

$$E_{AB}=:\partial_AV_B-\partial_BV_A(-1)^{\varepsilon_A\varepsilon_B}, \tag{3.6}$$

$$S=:\mathcal{L}-V_ADZ^A(-1)^{\varepsilon_A}=(1-N_{DZ})\mathcal{L}, \tag{3.7}$$

$$N_{DZ}=:DZ^A\frac{\partial}{\partial(DZ^A)}\,. \tag{3.8}$$

4. Functional and local master equations

Let the metric (3.6) be invertible, and E^{AB} be its inverse,

$$E_{AB}E^{BC}=\delta^C_A. \tag{4.1}$$

Let $F(Z,DZ), G(Z,DZ)$ be two arbitrary local functions; their local antibracket is defined as

$$(F,G)(Z,DZ)=:F(Z,DZ)\overleftarrow{\partial}_AE^{AB}(Z,DZ)\overrightarrow{\partial}_BG(Z,DZ), \tag{4.2}$$

in terms of explicit Z^A-derivatives ∂_A.

In turn, let $F[Z],G[Z]$ be two arbitrary functionals; their functional antibracket is defined as

$$(F,G)'[Z]=:F[Z]\int d\mu(\Gamma)\frac{\overleftarrow{\delta}}{\delta Z^A}E^{AB}(Z,DZ)\frac{\overrightarrow{\delta}}{\delta Z^B}G[Z]. \tag{4.3}$$

Due to the definition of the V_A, the second in (3.3), together with the definition (3.6) of the E_{AB}, each of the antibrackets, (4.2) and (4.3), satisfies its polarized Jacobi identity,[1]

$$\begin{aligned}&((Z^A,Z^B),Z^C)(-1)^{(\varepsilon_A+1)(\varepsilon_C+1)}\\&\quad+\text{cyclic perm.}(A,B,C)=0,\end{aligned} \tag{4.4}$$

$$\begin{aligned}&((Z^A(\Gamma),Z^B(\Gamma'))',Z^C(\Gamma''))'(-1)^{(\varepsilon_A+1)(\varepsilon_C+1)}\\&\quad+\text{cyclic perm.}(A,\Gamma;B,\Gamma';C,\Gamma'')=0.\end{aligned} \tag{4.5}$$

Now, we are in a position to show that the functional master equation

$$\frac{1}{2}(\Sigma,\Sigma)'=0, \tag{4.6}$$

is satisfied as for the action $\Sigma[Z]$ (4.2), provided that the local master equation

$$\frac{1}{2}(S,S)=0, \tag{4.7}$$

is satisfied as for the function $S(Z,DZ)$ (3.7). Indeed, by inserting the functional derivative (3.5) into the left-hand side of the equation (4.6), we have

$$\frac{1}{2}(\Sigma,\Sigma)'=\int d\mu(\Gamma)\left(\frac{1}{2}(S,S)+D\mathcal{L}\right)=0, \tag{4.8}$$

due to the local master equation (4.7), together with the boundary condition,

$$\mathcal{L}|_{\partial\Gamma}=0. \tag{4.9}$$

As a simple example of the Jacobi identity as for the functional antibracket (4.3), consider only the first equality in the formula (4.8), before the use of the equation (4.7),

$$(\Sigma,\Sigma)'=\int d\mu(\Gamma)(S,S), \tag{4.10}$$

which is valid for any functional of the form (3.2), with S defined by (3.7). Then, we get the non-polarized form of the functional Jacobi identity,

$$\begin{aligned}&((\Sigma,\Sigma)',\Sigma)'\\&\quad=\int d\mu(\Gamma)(S,S)\left(\frac{\overleftarrow{\partial}}{\partial Z^A}+\frac{\overleftarrow{\partial}}{\partial(DZ^A)}D\right)\left(DZ^A+(S,Z^A)\right)(-1)^{\varepsilon_A}\\&\quad=0.\end{aligned} \tag{4.11}$$

In fact, the second equality in eq. (4.11) holds due to the polarized functional Jacobi identity (4.5) derived in Appendix A. The integrand in eq. (4.11) demonstrates explicitly the structure of the original terms characteristic for the non-polarized functional Jacobi identity. Among other formal properties involved, the simplest one is the nilpotency of the de Rham's differential D, the second in the conditions (2.2), as well as the appearance of the boundary terms of the form "D(anything)", and the local non-polarized Jacobi identity,

$$((S,S),S)=0, \tag{4.12}$$

together with its (DZ)–dual identity,

$$(S,S)\frac{\overleftarrow{\partial}}{\partial(DZ^A)}D(Z^A,S)=0. \tag{4.13}$$

Finally, if we introduce the nilpotent functional odd Laplacian,

$$\Delta'=:\frac{1}{2}(\rho'[Z])^{-1}\int d\mu(\Gamma)(-1)^{\varepsilon_A}\frac{\delta}{\delta Z^A}\rho'[Z]E^{AB}(Z,DZ)\frac{\delta}{\delta Z^B}, \tag{4.14}$$

with $\rho'[Z]$ being a local functional measure,

[1] Here in (4.5), Γ' and Γ'' mean (X',C') and (X'',C'') as to stand for (X,C) in (2.1). Not to be confused with (3.1).