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# Periodic Airy process and equilibrium dynamics of edge fermions in a trap

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## HIGHLIGHTS

- We introduce a new stochastic process, the periodic Airy-2 process.
- It describes the imaginary-time dynamics of trapped fermions at finite temperature.
- New results for the real time dynamics of trapped fermions at finite temperature.

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## ABSTRACT

We establish an exact mapping between (i) the equilibrium (imaginary time) dynamics of non-interacting fermions trapped in a harmonic potential at temperature $T = 1/\beta$ and (ii) non-intersecting Ornstein–Uhlenbeck (OU) particles constrained to return to their initial positions after time $\beta$. Exploiting the determinantal structure of the process we compute the universal correlation functions both in the bulk and at the edge of the trapped Fermi gas. The latter corresponds to the top path of the non-intersecting OU particles, and leads us to introduce and study the time-periodic $\text{Airy}_2$ process, $\mathcal{A}_2^b(u)$, depending on a single parameter, the period $b$. The standard $\text{Airy}_2$ process is recovered for $b = +\infty$. We discuss applications of our results to the real time quantum dynamics of trapped fermions.



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## 1. Introduction

The Airy$_2$ process was introduced in Ref. [1] in the context of the discrete space, continuous time polynuclear growth model. Since then this process has appeared in many different contexts, such as directed last passage percolation [2], Dyson's Brownian motion [3], non-intersecting Brownian bridges and excursions (watermelons) [4,5], growth models [6], random tilings [7], interacting particle transport [8] and in the continuum KPZ equation [9] (for a review see [10,11]). To understand this process one can consider the following simple example. Consider a single Ornstein–Uhlenbeck (OU) process [12] where the position of a particle $x(\tau)$ in a one-dimensional harmonic potential evolves by the Langevin equation:

$$\frac{dx(\tau)}{d\tau} = -\mu_0\, x(\tau) + \eta(\tau), \tag{1}$$

where $\eta(\tau)$ is a centered Gaussian white noise, with correlator $\overline{\eta(\tau)\eta(\tau')} = \delta(\tau-\tau')$. We now consider an OU bridge, i.e., condition the OU process to start at $x = 0$ at $\tau \to -\infty$ and end at $x = 0$ at $\tau \to +\infty$. Suppose that we want to compute the probability density function (PDF) of the position of the walker at any fixed finite $\tau$. This probability can be computed by splitting the path into a left and a right part, calculating the probability of each part and then taking the product. For the left part the walker starts at $x = 0$ at time $-\infty$ hence at time $\tau$ its probability to be at $x$ becomes independent of $\tau$ and is given by

$$P_{\text{left}}(x,\tau) \propto e^{-\frac{\mu_0}{2}x^2}. \tag{2}$$

For the right part, we can consider the time-reversed path, which is just an independent but (statistically) identical copy of the left path. Hence the probability to reach $x$ at time $\tau$ from the right is given by

$$P_{\text{right}}(x,\tau) \propto e^{-\frac{\mu_0}{2}x^2}. \tag{3}$$

Taking the product and normalizing to unity, we get the PDF of the position of this OU bridge as [13]

$$P_{\text{stat}}(x) = \sqrt{\frac{\mu_0}{\pi}}\, e^{-\mu_0 x^2}, \tag{4}$$

where the subscript "stat" indicates that this PDF is stationary, i.e., independent of time $\tau$. In fact, this OU bridge process can also be identified to a time-periodic OU process (to be elaborated later in the paper) with period infinity.

We now consider an $N$-body generalization of this process, i.e., $N$ OU bridges, conditioned not to cross each other, $x_1(\tau) > x_2(\tau) > \cdots > x_N(\tau)$, at any time $\tau$. As in the $N = 1$ case, we assume that they started close to the origin (respecting the above ordering) at time $\tau \to -\infty$ and are conditioned to return to the same initial positions at time $\tau \to +\infty$. Again we can make the left–right decomposition of the paths at any intermediate time $\tau$. For the left part, the probability density that the $N$ walkers reach $x_1, x_2, \ldots, x_N$ at time $\tau$ again becomes independent of $\tau$ and is given by (see for instance [14])

$$P_{\text{left}}(x_1,\ldots,x_N,\tau) \propto e^{-\frac{\mu_0}{2}\sum_{j=1}^N x_j^2} \prod_{1\le j<i\le N} (x_i - x_j). \tag{5}$$

Incidentally this is also the joint PDF (JPDF) of the $N$ eigenvalues of an $N\times N$ real, symmetric Gaussian random matrix [known as the Gaussian Orthogonal Ensemble (GOE) [15]]. Similarly, for the right part, considering the time reversed trajectories, we get

$$P_{\text{right}}(x_1,\ldots,x_N,\tau) \propto e^{-\frac{\mu_0}{2}\sum_{j=1}^N x_j^2} \prod_{1\le j<i\le N} (x_i - x_j). \tag{6}$$

Hence, taking the product, we obtain the JPDF of these non-intersecting OU bridges as

$$P_{\text{stat}}(x_1,\ldots,x_N) = A_N e^{-\mu_0\sum_{j=1}^N x_j^2} \prod_{j=1}^N (x_i - x_j)^2 \tag{7}$$